Article

An Identity Involving Bell & Stirling Numbers

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Abstract

We apply the Euler's operator $(x\frac{d}{dx})^m$ to Dobinski's formula to show that $\sum_{j=0}^{n} j^m\, S_n^{[j]}$ is a linear combination of Bell numbers.

Keywords: Stirling numbers, Euler operator, Dobinski's formula, Bell numbers.

1. Introduction

We have the Dobinski's formula [1- 4]:

$$\sum_{j=0}^{n} S_n^{[j]}\, x^j = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!}\, x^k \,, \tag{1}$$

where $S_n^{[j]}$ represents a Stirling number of the second kind [2, 5]. If we employ the Euler's operator [2, 4] to determine $[\left(x\frac{d}{dx}\right) eq.(1)\,]_{x=1}$ we obtain:

$$\sum_{j=0}^{n} j\ S_n^{[j]} = B(n+1) - B(n), \tag{2}$$

with the presence of Bell numbers [2, 6-8]:

$$B(n) \equiv \sum_{j=0}^{n} S_n^{[j]} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{n!} \,. \tag{3}$$

Similarly, $[\left(x\frac{d}{dx}\right)^2 eq.(1)\,]_{x=1}$ implies the relation:

$$\sum_{j=0}^{n} j^2\, S_n^{[j]} = B(n+2) - 2\, B(n+1), \tag{4}$$

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and $[\left(x\frac{d}{dx}\right)^3 eq.(1)]_{x=1}$ gives the identity:

$$\sum_{j=0}^{n} j^3\, S_n^{[j]} = B(n+3) - 3\,B(n+2) + B(n), \tag{5}$$

and so on. In Sec. 2 we apply the operator $(x\frac{d}{dx})^m$ to (1) to deduce an expression for $\sum_{j=0}^{n} j^m\ S_n^{[j]}$ as a linear combination of Bell numbers [7].

2. Euler's operator

We know the property [2- 4, 9, 10]:

$$(x\frac{d}{dx})^m f(x) = \sum_{k=0}^{m} S_m^{[k]}\ x^k\, f^{(k)}(x), \tag{6}$$

hence $[\left(x\frac{d}{dx}\right)^m eq.(1)]_{x=1}$ implies the interesting expression:

$$\sum_{j=0}^{n} j^m\ S_n^{[j]} = \sum_{k=0}^{n} \binom{n}{k} \sum_{r=0}^{m} r!\ S_m^{[r]}\, S_{n-k}^{[r]}\, B(k), \tag{7}$$

which is a linear combination of Bell numbers. For $m = 0$ we recover (3); if $m = 1$ then (2) and (7) give the relation [2]:

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k). \tag{8}$$

For the case $m = 2$ the identities (4), (7), $S_n^{[1]} = 1$ and $S_n^{[2]} = 2^{n-1} - 1,\ n \geq 1$ imply the formula:

$$\sum_{k=0}^{n-1} \binom{n}{k} \frac{B(k)}{2^k} = \frac{1}{2^n}[B(n+2) - B(n+1) - B(n)]. \tag{9}$$

Similarly, if $m = 3$ then from (5) and (7):

$$\sum_{k=0}^{n-1} \binom{n}{k} \frac{B(k)}{3^k} = \frac{1}{3^n}[B(n+3) - 3\,B(n+2) + 2\,B(n+1) - B(n)]. \tag{10}$$

Spivey [11] obtained the following expression:

$$\sum_{k=0}^{n} (-1)^k\, k^m\, S_n^{(k)} = \sum_{j=0}^{m} (-1)^j\ j!\ S_m^{[j]}\, S_{n+1}^{(j+1)}\,, \tag{11}$$

involving the Stirling numbers of the first kind, which can be seen as companion of (7), and for $m = 1$ gives the known relation for the harmonic numbers [2, 5, 12]:

$$H_n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k \, k \, S_n^{(k)} = \frac{(-1)^{n+1}}{n!} S_{n+1}^{(2)} \,. \tag{12}$$

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