Article

Paule-Schneider's Identities for Harmonic Numbers

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Abstract

Paule-Schneider applied the Mathematica package *Sigma* and the Zeilberger's algorithm to find interesting identities for harmonic numbers; here we exhibit elementary proofs for some of those identities.

Keywords: Harmonic numbers, Zeilberger's algorithm.

1. Introduction

Paule-Schneider [1] used the Mathematica package *Sigma* [2] and the Zeilberger's algorithm [3, 4] to find the following identities for harmonic numbers [5, 6]:

$$\sum_{j=1}^{n} H_j \binom{n}{j} = 2^n \left[H_n - \sum_{j=1}^{n} \frac{1}{j\,2^j} \right], \qquad H_n \equiv \sum_{k=1}^{n} \frac{1}{k}, \tag{1}$$

$$\sum_{j=1}^{n} j\, H_j \binom{n}{j} = -\frac{1}{2} + 2^{n-1} \left[1 + n\, H_n - n \sum_{j=1}^{n} \frac{1}{j\,2^j} \right], \tag{2}$$

$$\sum_{j=1}^{n} H_j \binom{n}{j}^2 = \left[2\, H_n - H_{2n} \right] \binom{2n}{n}, \tag{3}$$

$$\sum_{j=1}^{n} j\, H_j \binom{n}{j}^2 = \frac{1}{4} \left[1 + 4n\, H_n - 2n\, H_{2n} \right] \binom{2n}{n}. \tag{4}$$

Here we employ known relations to give elementary proofs of these interesting identities.

2. Proofs of (1) and (2)

We have the expressions [7]:

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$$H_j = \sum_{q=1}^{j} \binom{j}{q} \frac{(-1)^{q+1}}{q}, \tag{5}$$

$$\binom{n}{j}\binom{j}{q} = \binom{n}{q}\binom{n-q}{j-q}, \qquad \sum_{j=q}^{n} \binom{n-q}{j-q} = 2^{n-q}, \tag{6}$$

and [8]:

$$\sum_{q=1}^{n} \binom{n}{q} \frac{(x-1)^q}{q} = \sum_{q=1}^{n} \frac{x^q}{q} - H_n \quad \overset{x=\frac{1}{2}}{\longrightarrow} \quad \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{q \cdot 2^q} = H_n - \sum_{q=1}^{n} \frac{1}{q \cdot 2^q}, \tag{7}$$

then:

$$\sum_{j=1}^{n} H_j \binom{n}{j} \overset{(5)}{=} \sum_{q=1}^{n} \frac{(-1)^{q+1}}{q} \sum_{j=q}^{n} \binom{n}{j}\binom{j}{q} \overset{(6)}{=} \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{q} \sum_{j=q}^{n} \binom{n-q}{j-q},$$

$$\overset{(6)}{=} 2^n \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{q \cdot 2^q} \overset{(7)}{=} \text{identity (1), q.e.d.}$$

It is simple to show the relations:

$$\sum_{j=q}^{n} j \binom{n-q}{j-q} = \frac{1}{2} (n+q)\, 2^{n-q}, \tag{8}$$

$$\sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{2^q} = 1 - \frac{1}{2^n}, \tag{9}$$

therefore:

$$\sum_{j=1}^{n} j\, H_j \binom{n}{j} \overset{(5,6)}{=} \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{q} \sum_{j=q}^{n} j \binom{n-q}{j-q},$$

$$\overset{(8)}{=} 2^{n-1} \left[n \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{q \cdot 2^q} + \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{2^q} \right] \overset{(7,9)}{=} \text{identity (2), q.e.d.}$$

3. Proof of (3)

We know the expressions [7]:

$$\sum_{k=0}^{n-q} \binom{n-q}{k}\binom{y}{k+q} = \binom{n-q+y}{n} \overset{y=n}{\longrightarrow} \sum_{k=0}^{n-q} \binom{n-q}{k}\binom{n}{k+q} = \binom{2n-q}{n}, \qquad (10)$$

$$\sum_{q=0}^{k} (-1)^q \binom{k}{q}\binom{y-q}{j} = \binom{y-k}{j-k} \underset{j=n}{\overset{y=2n}{\longrightarrow}} \sum_{q=0}^{k} (-1)^q \binom{k}{q}\binom{2n-q}{n} = \binom{2n-k}{n-k}, \qquad (11)$$

besides [9, 10]:

$$\frac{1}{q}\binom{n}{q} = \sum_{k=q}^{n} \frac{1}{k}\binom{k}{q}, \qquad (12)$$

and [11]:

$$(H_{2n} - H_n)\binom{2n}{n} = \sum_{r=n}^{2n-1} \frac{1}{2n-r}\binom{r}{n} = \sum_{k=1}^{n} \frac{1}{k}\binom{2n-k}{n-k}, \qquad (13)$$

thus:

$$\sum_{j=1}^{n} H_j \binom{n}{j}^2 \overset{(5)}{=} \sum_{q=1}^{n} \frac{(-1)^{q+1}}{q} \sum_{j=q}^{n} \binom{n}{j}\binom{n}{j}\binom{j}{q} \overset{(6)}{=} \sum_{q=1}^{n} \binom{n}{q} \frac{(-1)^{q+1}}{q} \sum_{k=0}^{n-q} \binom{n-q}{k}\binom{n}{k+q},$$

$$\overset{(10)}{=} \sum_{q=1}^{n} \frac{(-1)^{q+1}}{q} \binom{n}{q}\binom{2n-q}{n} \overset{(12)}{=} \sum_{k=1}^{n} \frac{1}{k} \sum_{q=1}^{k} (-1)^{q+1} \binom{k}{q}\binom{2n-q}{n},$$

$$= \sum_{k=1}^{n} \frac{1}{k}\left[\binom{2n}{n} - \sum_{q=0}^{k} (-1)^q \binom{k}{q}\binom{2n-q}{n}\right] = H_n \binom{2n}{n} - \sum_{k=1}^{n} \frac{1}{k} \sum_{q=0}^{k} (-1)^q \binom{k}{q}\binom{2n-q}{n},$$

$$\overset{(11)}{=} \binom{2n}{n} H_n - \sum_{k=1}^{n} \frac{1}{k}\binom{2n-k}{n-k} \overset{(13)}{=} \text{identity (3), q.e.d.}$$

A similar process allows show (4) but we do not give here.

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References

1. P. Paule, C. Schneider, *Computer proofs of a new family of harmonic number identities*, Adv. Appl. Math. **31** (2003) 359-378.
2. C. Schneider, *An implementation of Karr's summation algorithm in Mathematica,* Sém. Lothar. Combin. S**43** b (2000) 1-10.
3. D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80**, No. 2 (1990) 207-211.
4. M. Petkovsek, H. S. Wilf, D. Zeilberger, *A=B*, *symbolic summation algorithms*, A. K. Peters, Wellesley, MA (1996).
5. J. López-Bonilla, R. López-Vázquez, *Harmonic numbers in terms of Stirling numbers of the second kind*, Prespacetime Journal **8**, No. 2 (2017) 233-234.
6. B. E. Carvajal-Gámez, J. López-Bonilla, R. López-Vázquez, *On harmonic numbers*, Prespacetime Journal **8**, No. 4 (2017) 484-489.
7. J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers*, World Scientific, Singapore (2016).
8. J. Riordan, *Combinatorial identities*, John Wiley & Sons, New York (1968).
9. C. H. Jones, *Generalized hockey stick identities and N-dimensional block walking,* Fibonacci Quart. **34**, No. 3 (1996) 280-288.
10. D. Zagier, *Curious and exotic identities for Bernoulli numbers,* in "Bernoulli numbers and zeta functions", Eds. T. Arakawa, T. Ibukiyama, M. Kaneko, Springer, Japan (2014) 239-262.
11. J. Spiess, *Some identities involving harmonic numbers,* Maths. of Comput. **55**, No. 192 (1990) 839-863.