Article

Churchill-Plebañski & Petrov Classifications for Spacetimes of Embedding Class One

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Abstract

For each Churchill-Plebañski type of the trace-free second fundamental form we obtain, via the Newman-Penrose technique, the corresponding Petrov type of the Weyl tensor.

Keywords: Local embedding, isometric embedding, second fundamental form, conformal tensor, Churchill-Plebañski classification.

1. Introduction

We consider spacetimes of class one, that is, the local and isometric embedding of R_4 into E_5, then the corresponding second fundamental form $b_{\mu\nu} = b_{\nu\mu}$ satisfies the Gauss-Codazzi equations [1-4]:

$$R_{\mu\nu\alpha\beta} = \varepsilon\left(b_{\mu\alpha}\, b_{\nu\beta} - b_{\mu\beta}\, b_{\nu\alpha}\right), \qquad b_{\mu\nu;\alpha} = b_{\mu\alpha;\nu}\,, \qquad \varepsilon = \pm 1, \tag{1}$$

for the the intrinsic and extrinsic properties of the 4-space.

The Weyl tensor is given by [1, 4]:

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{2}\left(R_{\mu\beta}\, g_{\nu\alpha} + R_{\nu\alpha}\, g_{\mu\beta} - R_{\mu\alpha}\, g_{\nu\beta} - R_{\nu\beta}\, g_{\mu\alpha}\right) + \frac{R}{6}\left(g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\beta}\, g_{\nu\alpha}\right),$$

(1)

$$= \varepsilon\left[B_{\mu\alpha}B_{\nu\beta} - B_{\mu\beta}B_{\nu\alpha} + \frac{1}{2}\left(A_{\mu\alpha}g_{\nu\beta} + A_{\nu\beta}g_{\mu\alpha} - A_{\mu\beta}g_{\nu\alpha} - A_{\nu\alpha}g_{\mu\beta}\right) - \frac{A}{6}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right)\right], \tag{2}$$

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where $B_{\alpha\beta} = b_{\alpha\beta} - \frac{b}{4}\, g_{\alpha\beta}$, $b = b^{\lambda}{}_{\lambda}$, $A_{\alpha\beta} = B_{\alpha}{}^{\mu} B_{\beta\mu}$ and $A = A^{\lambda}{}_{\lambda}$.

In Sec. 2 we employ the Newman-Penrose (NP) technique [4-7] to project (2) onto the null tetrad $(l^{\alpha}, n^{\alpha}, m^{\alpha}, \overline{m}^{\alpha})$, and thus to relate the Churchill-Plebañski types [8-12] of $B_{\mu\nu}$ with the Petrov types [4, 13-16] of $C_{\mu\nu\alpha\beta}$.

2. Petrov and Churchill-Plebañski classifications

We project (2) onto the null tetrad to obtain [17, 18]:

$$\psi_0 = 4\varepsilon\left(\Omega_{00}\,\Omega_{02} - \Omega_{01}{}^2\right), \qquad \psi_1 = 2\varepsilon(\Omega_{00}\,\Omega_{12} - 2\Omega_{01}\,\Omega_{11} + \Omega_{02}\,\overline{\Omega}_{01}),$$

$$\psi_2 = \frac{2}{3}\varepsilon\left(\Omega_{00}\,\Omega_{22} - 2\Omega_{01}\,\overline{\Omega}_{12} + \Omega_{02}\,\overline{\Omega}_{02} - 4\Omega_{11}{}^2 + 4\Omega_{12}\,\overline{\Omega}_{01}\right), \tag{3}$$

$$\psi_3 = 2\varepsilon(\overline{\Omega}_{02}\,\Omega_{12} - 2\overline{\Omega}_{12}\,\Omega_{11} + \Omega_{22}\,\overline{\Omega}_{01}), \qquad \psi_4 = 4\varepsilon\left(\overline{\Omega}_{02}\,\Omega_{22} - \overline{\Omega}_{12}{}^2\right),$$

where ψ_r and Ω_{ab} are the NP components of $C_{\mu\nu\alpha\beta}$ and $B_{\mu\nu}$, respectively [4-7, 19, 20].

Now for each Churchill-Plebañski type of $B_{\alpha\beta}$ [18] we shall determine the Petrov type of (2) via the following algorithm developed in [21]:

$$\begin{array}{ccccccc}
 & \text{yes} & & \text{no} & & \text{yes} & \\
O & \longleftarrow & \psi_a = 0, \quad a = 0, \dots, 4 & \longrightarrow & G_b = 0, \quad b = 0, \dots, 5 & \longrightarrow & N \\
 & & & & \downarrow \text{ no} & & \\
 & \text{no} & & \text{no} & \text{yes} & & \\
I & \longleftarrow & K^3 = 27J^2 & \longleftarrow & J = K = 0 & \longrightarrow & III \\
 & & \downarrow \text{ yes} & & & & \\
 & \text{no} & G_c + \lambda\psi_c = 0, \quad c = 0,1,3,4 & \text{yes} & & & \\
II & \longleftarrow & & \longrightarrow & D & & \\
 & & G_2 + 2G_5 + 3\lambda\psi_2 = 0, & & & &
\end{array} \tag{4}$$

such that:

$$G_0 = 2(\psi_0\psi_2 - {\psi_1}^2)\,, \quad G_1 = \psi_0\psi_3 - \psi_1\psi_2\,, \quad G_2 = {\psi_2}^2 + \psi_0\psi_4 - 2\psi_1\psi_3\,,$$

$$G_3 = \psi_1\psi_4 - \psi_2\psi_3\,, \quad G_4 = 2(\psi_2\psi_4 - {\psi_3}^2)\,, \quad G_5 = 2(\psi_1\psi_3 - {\psi_2}^2)\,, \tag{5}$$

$$J = -\psi_3 G_1 + \tfrac{1}{2}(\psi_2 G_5 + \psi_4 G_0)\,, \quad K = G_2 - G_5\,, \quad \lambda^2 = \tfrac{K}{3}\,, \quad \lambda^3 = -J\,.$$

This method (4) is used by Differential Geometry (Maple software package) [22] to know the Petrov type of a tensor with the algebraic symmetries of the Weyl tensor.

McIntosh et al [18] give a canonical set of nonzero NP quantities Ω_{ab} for each Churchill-Plebañski type, then for them we can determine the corresponding Petrov types of (2) using (3)-(5):

a$_1$). $\boldsymbol{\Omega_{00} = \Omega_{22},\ \Omega_{02} = \bar{\Omega}_{02},\ \Omega_{11}}$: $\boldsymbol{[T - S_1 - S_2 - S_3]_{[1-1-1-1]}}$.

In this case from (3) and (5) we have the values:

$$\psi_0 = \psi_4 = 4\varepsilon\Omega_{00}\Omega_{02}\,,\ \psi_1 = \psi_3 = 0\,,\ \psi_2 = \tfrac{2}{3}\varepsilon({\Omega_{00}}^2 + {\Omega_{02}}^2 - 4{\Omega_{11}}^2), G_0 = G_4 = 2\psi_0\psi_2\,,$$

$$G_1 = G_3 = 0\,,\ G_2 = {\psi_0}^2 + {\psi_2}^2\,,\ G_5 = -2{\psi_2}^2\,,\ J = \psi_2({\psi_0}^2 - {\psi_2}^2), \quad K = {\psi_0}^2 + 3{\psi_2}^2\,,$$

with $K^3 \neq 27J^2$, then the algorithm (4) implies that $C_{\mu\nu\alpha\beta}$ has Petrov type I.

a$_2$). $\boldsymbol{\Omega_{02} = \bar{\Omega}_{02},\ \Omega_{11}}$: $\boldsymbol{[2T - S_1 - S_2]_{[1-1-1]}}$.

Therefore $\psi_r = 0,\ r \neq 2,\ \ \psi_2 = \frac{2}{3}\varepsilon\,({\Omega_{02}}^2 - 4{\Omega_{11}}^2)$ and:

$$G_a = 0,\ a \neq 2,5\,, \quad G_5 = -2G_2 = -2{\psi_2}^2\,, \quad \lambda = \psi_2\,, \quad K = 2{\psi_2}^2\,, \quad J = -{\psi_2}^3\,, \tag{6}$$

hence (4) gives type D for (2).

a$_3$). $\boldsymbol{\Omega_{00} = \Omega_{22},\ \Omega_{11}}$: $\boldsymbol{[T - 2S_1 - S_2]_{[1-1-1]}}$.

Then $\psi_b = 0,\ b \neq 2,\ \psi_2 = \frac{2}{3}\varepsilon\,({\Omega_{00}}^2 - 4{\Omega_{11}}^2)$ with the same expressions (6), thus $C_{\mu\nu\alpha\beta}$ has Petrov type D.

a$_4$). $\mathbf{\Omega_{11}}$: $[\mathbf{2T-2S}]_{[\mathbf{1-1}]}$.

Now $\psi_c = 0$, $c \neq 2$, $\psi_2 = -\frac{8}{3}\varepsilon\,{\Omega_{11}}^2$ and (6) are valid again, therefore the Weyl tensor has type D.

a$_5$). $\mathbf{\Omega_{02} = 2\Omega_{11}}$: $[\mathbf{3T-S}]_{[\mathbf{1-1}]}$: $C_{\mu\nu\alpha\beta} = 0$ because $\psi_r = 0$, $r = 0, \dots, 4$, then (4) gives type O.

a$_6$). $\mathbf{\Omega_{00} = \Omega_{22} = 2\Omega_{11}}$: $[\mathbf{T-3S}]_{[\mathbf{1-1}]}$: From (3) we obtain $\psi_a = 0$, $\forall\, a$, hence (2) has type O.

a$_7$). $\mathbf{\Omega_{ab} = 0}$, $\forall\, \boldsymbol{a}, \boldsymbol{b}$: $[\mathbf{4T}]_{[\mathbf{1}]}$: Evidently the conformal tensor is type O.

a$_8$). $\mathbf{\Omega_{00} = -\Omega_{22}}$, $\mathbf{\Omega_{11}}$, $\mathbf{\Omega_{02} = \bar{\Omega}_{02}}$: $[\boldsymbol{Z} - \bar{\boldsymbol{Z}} - \boldsymbol{S_1} - \boldsymbol{S_2}]_{[\mathbf{1-1-1-1}]}$.

Now we have:

$$\psi_0 = -\psi_4 = 4\varepsilon\Omega_{00}\Omega_{02},\ \psi_1 = \psi_3 = 0, \psi_2 = \frac{2}{3}\varepsilon\left({\Omega_{02}}^2 - {\Omega_{00}}^2 - 4{\Omega_{11}}^2\right), G_0 = -G_4 = 2\psi_0\psi_2,$$

$$G_1 = G_3 = 0, G_2 = {\psi_2}^2 - {\psi_0}^2, G_5 = -2{\psi_2}^2, J = -\psi_2\left({\psi_0}^2 + {\psi_2}^2\right), K = -{\psi_0}^2 + 3{\psi_2}^2,\ K^3 \neq 27J^2,$$

and the algorithm (4) implies type I for $C_{\mu\nu\alpha\beta}$.

a$_9$). $\mathbf{\Omega_{00} = -\Omega_{22}}$, $\mathbf{\Omega_{11}}$: $[\boldsymbol{Z} - \bar{\boldsymbol{Z}} - \mathbf{2}\boldsymbol{S}]_{[\mathbf{1-1-1}]}$.

Here $\psi_b = 0$, $b \neq 2$, $\psi_2 = -\frac{2}{3}\varepsilon({\Omega_{00}}^2 + 4{\Omega_{11}}^2)$ and (6), therefore (2) has Petrov type D.

a$_{10}$). $\mathbf{\Omega_{02} = \bar{\Omega}_{02}}$, $\mathbf{\Omega_{11}}$, $\mathbf{\Omega_{22}}$: $[\mathbf{2}\boldsymbol{N} - \boldsymbol{S_1} - \boldsymbol{S_2}]_{[\mathbf{2-1-1}]}$.

In this case from (3) and (5):

$$\psi_r = 0,\ r \neq 2, 4, \quad \psi_2 = \frac{2}{3}\varepsilon\left({\Omega_{02}}^2 - 4{\Omega_{11}}^2\right), \quad \psi_4 = 4\varepsilon\,\Omega_{02}\Omega_{22}, \quad G_a = 0,\ a \neq 2, 5,$$

$$G_5 = -2G_2 = -2{\psi_2}^2, \quad J = -{\psi_2}^3, \quad K = 3{\psi_2}^2, \quad K^3 = 27J^2, \quad \lambda = \psi_2,$$

but $G_4 + \lambda\psi_4 \neq 0$, hence the Weyl tensor has type II.

a_{11}). $\mathbf{\Omega_{11}}$, $\mathbf{\Omega_{22}}$: $[\mathbf{2N - 2S}]_{[\mathbf{2-1}]}$.

From (3) we obtain $\psi_a = 0,\ a \neq 2,\ \ \psi_2 = -\frac{8}{3}\varepsilon\,\Omega_{11}{}^2$ with the relations (6), thus the algorithm (4) gives the type D.

a_{12}). $\mathbf{\Omega_{02} = 2\Omega_{11}}$, $\mathbf{\Omega_{22}}$: $[\mathbf{3N - S}]_{[\mathbf{2-1}]}$.

Now we have $\psi_b = 0,\ b \neq 4,\ \psi_4 = 4\varepsilon\,\Omega_{02}\Omega_{22}$, $G_r = 0,\ r = 0, \dots, 5$, therefore $C_{\mu\nu\alpha\beta}$ is type N.

a_{13}). $\mathbf{\Omega_{22}}$: $[\mathbf{4N}]_{[\mathbf{2}]}$: Then $\psi_r = 0,\ \forall\, r,$ hence (2) has Petrov type O.

a_{14}). $\mathbf{\Omega_{01} \neq \bar{\Omega}_{01}}$, $\mathbf{\Omega_{02} = 2\Omega_{11}}$: $[\mathbf{3N - S}]_{[\mathbf{3-1}]}$.

Here we deduce that $\psi_c = 0,\ c \neq 0, 1$, $G_a = 0,\ a \neq 0,\ K = J = 0,$ thus from (4) the conformal tensor is type III.

a_{15}). $\mathbf{\Omega_{01}}$: $[\mathbf{4N}]_{[\mathbf{3}]}$: In this case, $\psi_r = 0, r \neq 0,\ \psi_0 = -4\varepsilon\Omega_{01}{}^2,\ G_a = 0, \forall\, a$, therefore (4) gives the type N.

The analysis realized in a_1), … , a_{15}) is in agreement with the Table I of McIntosh et al [18] and the Table II of Barnes [23], which shows the utility of the algorithm (4) to obtain the Petrov type of a tensor with the same symmetries as the Weyl tensor. Our study is algebraic because is based in the Gauss equation (1) for the second fundamental form, but is necessary to employ the Codazzi differential equation and thus perhaps under certain conditions some Churchill-Plebañski or Petrov types may be prohibited for embedding class one, or is possible that the Codazzi's relation imposed constrains to the null congruences associated with the NP tetrad [24].

Received October 16, 2017; Accepted December 03, 2017

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