Article

A New Approach to Inextensible Flows of Curves with Ribbon Frame

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Abstract. In this paper, we study inextensible flows of curves according to Ribbon frame in $\mathbb{E}^3$. We research inextensible flows of curves according to Ribbon frame in $\mathbb{E}^3$. Necessary and sufficient conditions for an inelastic curve flow are expressed as a partial differential equation involving the curvature.

Keywords: Inextensible flows, ribbon frame, curvatures.

1 Introduction

The flow of a curve or surface is said to be inextensible if, in the former case, the arclength is preserved, and in the latter case, if the intrinsic curvature is preserved. Physically, inextensible curve and surface flows are characterized by the absence of any strain energy induced from themotion. Kwon investigated inextensible flows of curves and developable surfaces in $\mathbb{E}^3$. Necessary and sufficient conditions for an inextensible curve flow first expressed as a partial differential equation involving the curvature and torsion. Then, they derived the corresponding equations for the inextensible flow of a developable surface, and showed that it suffices to describe its evolution in terms of two inextensible curve flows, [9,10].

A ribbon is a surface swept out by a line segment turning as it moves along a central curve. For narrow magnetic ribbons, for which the length of the line segment is much less than the length of the curve, the anisotropy induced by the magnetostatic interaction is biaxial, with hard axis normal to the ribbon and easy axis along the central curve. The micromagnetic energy of a narrow ribbon reduces to that of a one dimensional ferromagnetic wire, but with curvature, torsion and local anisotropy modified by the rate of turning. These general results are applied to two examples, namely a helicoid ribbon, for which the central curve is a straight line, and a Möbius ribbon, for which the central curve is a circle about which the line segment executes a 180 twist. A repetitive crystal-like pattern is spontaneously formed upon the twisting of straight ribbons. Bohr and Markvorsen [3] gave a general description of developable ribbons using a ruled procedure where ribbons are uniquely described by two generating functions. This construction defines a differentiable frame, the ribbon frame, which does not have singular points, whereby we avoid the shortcomings of the Frenet–Serret frame. In [12] the author studied the focal curves according to Ribbon frame in the Euclidean 3-space $\mathbb{E}^3$.In this paper, we investigate inextensible flows of curves according to Ribbon frame in $\mathbb{E}^3$. Using the Ribbon frame of the given curve, we present partial differential equations. We give some characterizations for curvatures of a curve in $\mathbb{E}^3$.

2 Preliminaries

The center curve of a ribbon will be described by a unit speed parameterization $\gamma(s)$, and the ribbon itself will be described as a developable surface parametrization $\Upsilon(s,u)$ supported by the center curve in

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such a way that the center curve itself becomes a geodesic in the ribbon surface. The width of a ribbon is assumed to be constant and will in the following be denoted $2b$.

To construct any ribbon and hence its center curve, we will use two continuous functions, $w(s)$ and $\theta(s)$ defined in the interval $s \in [0, L]$, where L is the intrinsic length of the ribbon under construction. In our construction a key role is played by a unit vector field $\mathbf{A}(s)$, which is a field tangent to the ribbon and defined by having the angle $\theta(s)$ to the center curve of the ribbon. In fact, $\mathbf{A}(s)$ will be the direction field for the Darboux vector $\mathbf{D}(s)$ with the generating function $w(s)$ as a multiplying factor, i.e. $\mathbf{D}(s) = w(s)\mathbf{A}(s)$.

We let $\{\mathbf{e}(s), \mathbf{f}(s), \mathbf{g}(s)\}$ denote the unique orthonormal triple of vector solutions to the following differential system [3]:

$$\begin{aligned} \dot{\mathbf{e}}(s) &= w(s)\mathbf{A}(s) \times \mathbf{e}(s) \\ \dot{\mathbf{f}}(s) &= w(s)\mathbf{A}(s) \times \mathbf{f}(s)2.1 \\ \dot{\mathbf{g}}(s) &= w(s)\mathbf{A}(s) \times \mathbf{g}(s), \end{aligned} \tag{2.1}$$

where the unit vector field $\mathbf{A}(s)$ is defined in terms of $\mathbf{e}(s)$ and $\mathbf{g}(s)$ as follows:

$$\mathbf{A}(s) = \cos(\theta(s))\mathbf{e}(s) + \sin(\theta(s))\mathbf{g}(s), \tag{2.2}$$

and where – for uniqueness purposes – we also apply the following arbitrary initial conditions referring to a given fixed coordinate system and basis in $\mathbb{R}^3$

$$\{\mathbf{e}(0), \mathbf{f}(0), \mathbf{f}(0)\} = \{(1,0,0), (0,1,0), (0,0,1)\}. \tag{2.3}$$

The system (2.1) can be written explicitly as follows:

$$\begin{aligned} \dot{\mathbf{e}}(s) &= w(s)\sin(\theta(s))\mathbf{f}(s) \\ \dot{\mathbf{f}}(s) &= -w(s)\sin(\theta(s))\mathbf{e}(s) + w(s)\cos(\theta(s))\mathbf{g}(s)2.4 \\ \dot{\mathbf{g}}(s) &= -w(s)\cos(\theta(s))\mathbf{f}(s), \end{aligned} \tag{2.2}$$

where the dot notation means differentiation with respect to the unit speed parameter s. In compact matrix notation [3]:

$$\dot{\mathbf{R}}(s) = \mathbf{R}(s)\Xi(s), \tag{2.5}$$

where $\mathbf{R}(s)$ is the orthogonal matrix (with $2\det(\mathbf{R}(s)) = 1$) whose columns are the coordinate functions of $\{\mathbf{e}(s), \mathbf{f}(s)$, and $\mathbf{g}(s)\}$ respectively, and where

$$\Xi(s) = \begin{bmatrix} 0 & -w(s)\sin(\theta(s)) & 0 \\ w(s)\sin(\theta(s)) & 0 & -w(s)\cos(\theta(s)) \\ 0 & w(s)\cos(\theta(s)) & 0 \end{bmatrix}.$$

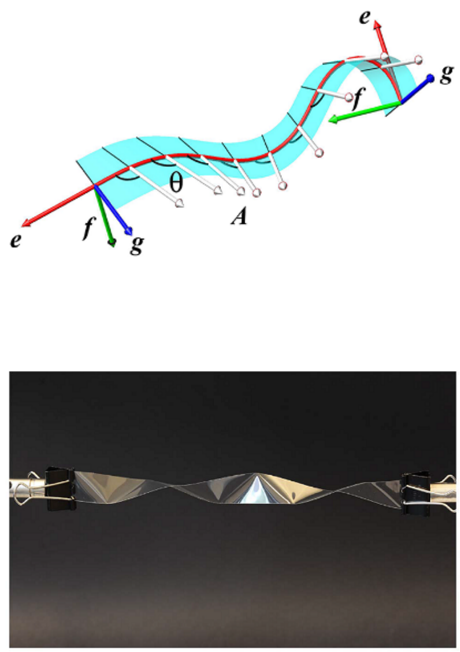

Figure 1: Ribbon with paper clips, [3].

3 Inextensible Flows of Curves According to Ribbon frame in $\mathbb{E}^3$

Throughout this article, we assume that $F : [0, l] \times [0, \omega] \to \mathbb{E}^3$ is a one parameter family of smooth curves in $\mathbb{E}^3$. Let u be the curve parametrization variable, $0 \leq u \leq l$.

The arclength of F is given by

$$s(u) =_0^u \left|\frac{\partial F}{\partial u}\right| du, \tag{3.1}$$

where

$$\left|\frac{\partial F}{\partial u}\right| = \left|\left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \right\rangle\right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

where $v = \left|\frac{\partial F}{\partial u}\right|$. The arclength parameter is $ds = vdu$.

Any flow of F can be represented as

$$\frac{\partial F}{\partial t} = k\mathbf{e} + l\mathbf{f} + m\mathbf{g}. \tag{3.3}$$

Letting the arclength variation be

$$s(u,t) =_0^u vdu. \tag{3.4}$$

In the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t) =_0^u \frac{\partial v}{\partial t}du = 0,$$

for all $u \in [0,l]$.

Definition 3.1. *A curve evolution* $F(u,t)$ *and its flow* $\frac{\partial F}{\partial t}$ *in* $\mathbb{E}^3$ *are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial F}{\partial u}\right| = 0. \tag{3.5}$$

Lemma 3.2. *Let* $\frac{\partial F}{\partial t}$ *be a smooth flow of the curve* F. *The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t} = \frac{\partial k}{\partial u} - lw\sin\theta, \tag{3.6}$$

Proof. By the formula of Ribbon, we have

$$\frac{\partial v}{\partial t} =< \mathbf{e}, (\frac{\partial k}{\partial u} - lw\sin\theta)\mathbf{e} + (lw\cos\theta + \frac{\partial m}{\partial u})\mathbf{g} + (kw\sin\theta + \frac{\partial l}{\partial u} - mw\cos\theta)\mathbf{f} >, \tag{3.7}$$

Since, we express Eq. (3.1), which proves the lemma.

Theorem 3.3. *Let* $\frac{\partial F}{\partial t}$ *be a smooth flow of the curve* F. *The flow is inextensible if and only if*

$$\frac{\partial k}{\partial u} = lw\sin\theta. \tag{3.8}$$

Proof. Now let $\frac{\partial F}{\partial t}$ be extensible. From Lemma 3.2, we have

$$\frac{\partial}{\partial t}s(u,t) =_0^u \frac{\partial v}{\partial t}du =_0^u (\frac{\partial k}{\partial u} - lw\sin\theta)du = 0, \tag{3.9}$$

Finally, we express the desired result.

We now restrict ourselves to arc length parametrized curves.

Lemma 3.4.

$$\begin{aligned} \frac{\partial \mathbf{e}}{\partial t} &= (kw\sin\theta + \frac{\partial l}{\partial s} - mw\cos\theta)\mathbf{f} + (lw\cos\theta + \frac{\partial m}{\partial s})\mathbf{g}, \\ \frac{\partial \mathbf{f}}{\partial t} &= -(kw\sin\theta + \frac{\partial l}{\partial s} - mw\cos\theta)\mathbf{e} + \psi\mathbf{g}, 3.10 \\ \frac{\partial \mathbf{g}}{\partial t} &= -(lw\cos\theta + \frac{\partial m}{\partial s})\mathbf{e} - \psi\mathbf{f}, \end{aligned} \tag{3.1}$$

where $\psi =< \frac{\partial \mathbf{f}}{\partial t}, \mathbf{g} >$.

Proof. Using definition of $\digamma$, we have

$$\frac{\partial \mathbf{e}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \digamma}{\partial s} = \frac{\partial}{\partial s}(k\mathbf{e} + l\mathbf{f} + m\mathbf{g}). \tag{3.11}$$

From the Ribbon equations, we have

$$\frac{\partial \mathbf{e}}{\partial t} = (\frac{\partial k}{\partial s} - lw\sin\theta)\mathbf{e} + (kw\sin\theta + \frac{\partial l}{\partial s} - mw\cos\theta)\mathbf{f} + (lw\cos\theta + \frac{\partial m}{\partial s})\mathbf{g}.$$

Using Lemma 3.2, we get

$$\frac{\partial \mathbf{e}}{\partial t} = (kw\sin\theta + \frac{\partial l}{\partial s} - mw\cos\theta)\mathbf{f} + (lw\cos\theta + \frac{\partial m}{\partial s})\mathbf{g}.$$

Using $< \frac{\partial \mathbf{f}}{\partial t}, \mathbf{f} > = < \frac{\partial \mathbf{g}}{\partial t}, \mathbf{g} > = 0$, we obtain

$$\begin{aligned} \frac{\partial \mathbf{f}}{\partial t} &= -(kw\sin\theta + \frac{\partial l}{\partial s} - mw\cos\theta)\mathbf{e} + \psi\mathbf{g}, \\ \frac{\partial \mathbf{g}}{\partial t} &= -(lw\cos\theta + \frac{\partial m}{\partial s})\mathbf{e} - \psi\mathbf{f}, \end{aligned}$$

where $\psi = < \frac{\partial \mathbf{f}}{\partial t}, \mathbf{g} >$.

The following theorem states the conditions on the curvature and torsion for the curve flow $\digamma(s,t)$ to be inextensible.

Theorem 3.5. *Let the flow* $\frac{\partial \digamma}{\partial t} = k\mathbf{e} + l\mathbf{f} + m\mathbf{g}$ *be inextensible. Then, the following system of partial differential equations holds:*

$$\frac{\partial}{\partial t}(w\sin\theta) = \frac{\partial}{\partial s}(kw\sin\theta) + \frac{\partial^2 l}{\partial s^2} - \frac{\partial}{\partial s}(mw\cos\theta) - lw^2\cos^2\theta - \frac{\partial m}{\partial s}w\cos\theta. \tag{3.12}$$

Proof. From Lemma 3.4, we have

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\partial \mathbf{e}}{\partial t} &= (\frac{\partial}{\partial s}(kw\sin\theta) + \frac{\partial^2 l}{\partial s^2} - \frac{\partial}{\partial s}(mw\cos\theta) - lw^2\cos^2\theta - \frac{\partial m}{\partial s}w\cos\theta)\mathbf{f} \\ &+ (-kw^2\sin^2\theta - \frac{\partial l}{\partial s}w\sin\theta + mw^2\cos\theta\sin\theta)\mathbf{e}3.13 \\ &+ (\frac{\partial}{\partial s}(lw\cos\theta) + \frac{\partial^2 m}{\partial s^2} + kw^2\sin\theta\cos\theta + \frac{\partial l}{\partial s}w\cos\theta - mw^2\cos^2\theta)\mathbf{g}. \end{aligned} \tag{3.2}$$

Also, from Ribbon frame we have

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{e}}{\partial s} = \frac{\partial(w\sin\theta)}{\partial t}\mathbf{f} - (kw^2\sin^2\theta + \frac{\partial l}{\partial s}w\sin\theta - mw^2\sin\theta\cos\theta)\mathbf{e} + (\psi w\sin\theta)\mathbf{g}. \tag{3.14}$$

This means that

$$\frac{\partial}{\partial t}(w\sin\theta) = \frac{\partial}{\partial s}(kw\sin\theta) + \frac{\partial^2 l}{\partial s^2} - \frac{\partial}{\partial s}(mw\cos\theta) - lw^2\cos^2\theta - \frac{\partial m}{\partial s}w\cos\theta,$$

Also,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{g}}{\partial t} &= (-\frac{\partial}{\partial s}(lw\cos\theta) - \frac{\partial^2 m}{\partial s^2} + \psi w \sin\theta)\mathbf{e}3.15 \\ &-(lw^2\cos\theta\sin\theta + \frac{\partial m}{\partial s}w\sin\theta + \frac{\partial \psi}{\partial s})\mathbf{f} \\ &-(\psi w\cos\theta)\mathbf{g}.\end{aligned} \tag{3.3}$$

Therefore,

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{g}}{\partial s} = -\frac{\partial(w\cos\theta)}{\partial t}\mathbf{f} + (kw^2\cos\theta\sin\theta - \frac{\partial l}{\partial s}w\cos\theta + mw^2\cos^2\theta)\mathbf{e} - (w\cos\theta\psi)\mathbf{g}. \tag{3.16}$$

Thus, we get

$$\frac{\partial(w\cos\theta)}{\partial t} = lw^2\cos\theta\sin\theta + \frac{\partial m}{\partial s}w\sin\theta + \frac{\partial\psi}{\partial s}.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{f}}{\partial t} &= (-(\frac{\partial}{\partial s}(kw\sin\theta) + \frac{\partial^2 l}{\partial s} - \frac{\partial}{\partial s}(mw\cos\theta)))\mathbf{e}3.17 \\ &+(-\psi w\cos\theta - k^2 w\sin^2\theta - \frac{\partial l}{\partial s}w\sin\theta + mw^2\cos\theta\sin\theta)\mathbf{f} + \frac{\partial\psi}{\partial s}\mathbf{g},\end{aligned} \tag{3.4}$$

Also, from Ribbon frame we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{f}}{\partial s} &= (-\frac{\partial(w\sin\theta)}{\partial t} - w^2 l\cos^2\theta + w\cos\theta\frac{\partial m}{\partial s})\mathbf{e} + (-w^2 k\sin^2\theta - w\sin\theta\frac{\partial l}{\partial s} \\ &+w^2 m\sin\theta\cos\theta - w\psi\cos\theta)\mathbf{f} + (-w^2 l\sin\theta\cos\theta - w\sin\theta\frac{\partial m}{\partial s} + \frac{\partial(w\cos\theta)}{\partial t})\mathbf{g}.\end{aligned}$$

This completes the proof.

Corollary 3.6.

$$\frac{\partial}{\partial s}(lw\cos\theta) + \frac{\partial^2 m}{\partial s^2} + \frac{\partial l}{\partial s}w\cos\theta = \psi w\sin\theta + mw^2\cos^2\theta - kw^2\sin\theta\cos\theta. \tag{3.19}$$

Corollary 3.7.

$$\frac{\partial}{\partial s}(kw\sin\theta) + \frac{\partial^2 l}{\partial s^2} - \frac{\partial}{\partial s}(mw\cos\theta) - lw^2\cos^2\theta - \frac{\partial m}{\partial s}w\cos\theta = lw^2\cos\theta\sin\theta + \frac{\partial m}{\partial s}w\sin\theta + \frac{\partial\psi}{\partial s}. \tag{3.20}$$

Corollary 3.8.

$$\frac{\partial\psi}{\partial s} = -w^2 l\sin\theta\cos\theta - w\sin\theta\frac{\partial m}{\partial s} + \frac{\partial}{\partial t}(w\cos\theta). \tag{3.21}$$

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