Article

On the Series Transformation Formula of Boyadzhiev

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Abstract

We exhibit an elementary deduction of Boyadzhiev's formula which turns power series into series of functions.

Keywords: Stirling numbers, Euler operator, Dobinski's relation, Bell numbers.

1. Introduction

Boyadzhiev [1, 2] obtained the expression:

$$Q \equiv \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{j=0}^{k} S_k^{[j]}\, x^j\, g^{(j)}(x) = \sum_{r=0}^{\infty} \frac{g^{(r)}(0)}{r!}\, f(r)\, x^r, \tag{1}$$

where $f(z)$ is an entire function, $S_k^{[j]}$ are the Stirling numbers of the second kind [3, 4], $g(z)$ is an analytic function in a region around the origin, and x belongs to this region. We observe that (1) turns power series into series of functions.

In Sec. 2 we give an elementary proof of (1) and we noted that it implies the identities of Quaintance-Gould [3] and Dobinski [3, 5, 6].

2. Boyadzhiev's formula

We know the following property satisfied by the Euler's operator $x\frac{d}{dx}$ [1-3, 6-10]:

$$(x\frac{d}{dx})^m\, h(x) = \sum_{j=0}^{m} S_m^{[j]}\, x^j\, h^{(j)}(x), \tag{2}$$

then:

$$Q \overset{(2)}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x\frac{d}{dx})^k\, g(x) = \sum_{r=0}^{\infty} \frac{g^{(r)}(0)}{r!} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x\frac{d}{dx})^k\, x^r, \tag{3}$$

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but from (2):

$$(x\frac{d}{dx})^k\, x^r = \sum_{j=0}^{k} S_k^{[j]}\, x^j\, \frac{d^j x^r}{dx^j} = r!\, \sum_{j=0}^{k} S_k^{[j]}\, \frac{x^r}{(r-j)!}\,,$$

thus (3) implies:

$$Q = \sum_{r=0}^{\infty} \frac{g^{(r)}(0)}{r!}\, x^r \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{j=0}^{k} \binom{r}{j} j!\; S_k^{[j]} = \sum_{r=0}^{\infty} \frac{g^{(r)}(0)}{r!}\, x^r \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, r^k, \qquad (4)$$

where was applied the relation [3, 11]:

$$\sum_{j=0}^{k} j!\, \binom{r}{j} S_k^{[j]} = r^k. \qquad (5)$$

The entire function $f(x)$ accepts expansion in Taylor's series, therefore $f(r) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, r^k$, hence (4) coincides with (1), q.e.d.

If in (1) we employ $g(x) = e^x$ and the expression [3]:

$$S_k^{[j]} = \frac{1}{j!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (j-r)^k\,, \qquad (6)$$

we obtain the identity of Quaintance-Gould [3]:

$$\sum_{j=0}^{n} \frac{x^j}{j!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} f(j-r) = e^{-x} \sum_{k=0}^{\infty} \frac{f(k)}{k!}\, x^k\,, \quad \forall\, x, \qquad (7)$$

where $f(x)$ is a polynomial of degree n. For the special case $f(x) = x^n$ the result (7) gives the Dobinski's formula [3, 5, 6]:

$$e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!}\, x^k = \sum_{j=0}^{n} S_n^{[j]}\, x^j\,, \qquad (8)$$

which for $x = 1$ implies the known relation for the Bell numbers [3, 12-14]:

$$B(n) \equiv \sum_{j=0}^{n} S_n^{[j]} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}\,.$$

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