Essay

Generalization of Riemann Zeta to Dedekind Zeta & Adelic Physics

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Abstract

Adelic physics postulates that a hierarchy of extensions of rationals defines an evolutionary hierarchy of physics of matter and cognition. The Galois groups of the extension play a key role in the proposal and define number theoretical symmetries. So called L-functions, in particular Riemann zeta coding information about ordinary primes and its generalizations to extensions of rationals are expected to play key role in adelic physics. So called radial conformal weights for the representations of super-symplectic algebra at boundaries of causal diamonds are proposed to correspond to zeros of ζ, and the discrete p-adic coupling constant evolution for the inverse of electroweak $U(1)$ coupling is proposed to correspond to the zeros of ζ associated to certain p-adic length scales for primes p near prime powers of 2. If this picture generalizes to all extensions K/Q of rationals, the zeta functions ζ_K would give quite an impressive quantitative grasp to adelic physics. In this article this possibility will be considered in more detail and it is found that extensions of rationals have a nice physical interpretation.

1 Introduction

I have considered adelic physics in [5] [7] as a fusion of physics of matter and cognition and characterized by a hierarchy of extensions of rationals inducing finite-dimensional extensions of p-adic numbers. A further insight to adelic physics comes from the possible physical interpretation of the L-functions appearing also in Langlands program [4]. The most important L-function would be generalization of Riemann zeta to extension of rationals. I have proposed several roles for ζ, which would be the simplest L-function assignable to rational primes, and for its zeros.

1. Riemann zeta itself could be identifiable as an analog of partition function for a system with energies given by logarithms of prime. One can define also the fermionic counterpart of ζ as ζ_F. In ZEO this function could be regarded as complex square root of thermodynamical partition function in accordance with the interpretation of quantum theory as complex square root of thermodynamics.

2. The zeros of zeta could define the conformal weights for the generators of super-symplectic algebra so that the number of generators would be infinite. The rough idea - certainly not correct as such except at the limit of infinitely large CD - is that the scaling operator $L_0 = r_M d/dr_M$, where r_M is light-like coordinate of light-cone boundary (containing upper or lower boundary of the causal diamond (CD)), has as eigenfunctions the functions $(r_M/r_0)^{s_n}$ $s_n = 1/2 + iy_n$, where s_n is the radial conformal weight identified as complex zero of ζ. Periodic boundary conditions for CD do not allow all possible zeros as conformal weights so that for given CD only finite subset corresponds to generators of the supersymplectic algebra. Conformal confinement would hold true in the sense that the sum $\sum_n s_n$ for physical states would be integer. Roots and their conjugates should appear as pairs in physical states.

3. On basis of numerical evidence Dyson [1] (`http://tinyurl.com/hjbfsuv`) has conjectured that the Fourier transform for the set formed by zeros of zeta consists of primes so that one could regard zeros as one-dimensional quasi-crystal. This hypothesis makes sense if the zeros of zeta decompose into disjoint sets such that each set corresponds to its own prime (and its powers) and one has $p^{iy} = U_{m/n} = exp(i2\pi m/n)$ (see the appendix of[6]). This hypothesis is also motivated by number theoretical universality [3, 5].

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4. I have considered the possibility [2] that the discrete values for the inverse of the electro-weak $U(1)$ coupling constant for a gauge field assignable to the Kähler form of CP_2 assignable to p-adic coupling constant evolution corresponds to poles of the fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ coming from $s_n/2$ (denominator) and pole at $s = 1$ (numerator) zeros of zeta assignable to rational primes. Note that also odd negative integers at real axis would be poles.

   It is also possible to consider scaling of the argument of $\zeta_F(s)$. More general coupling constant evolutions could correspond to $\zeta_F(m(s))$, where $m(s) = (as + b)/(cs + d)$ is Möbius transformation performed for the argument mapping upper complex plane to itself so that a, b, c, d are real and also rational by number theoretical universality.

2 Dedekind zeta function and TGD

Suppose for a moment that more precise formulations of these physics inspired conjectures make sense and even that their generalization for extensions K/Q of rationals holds true. This would solve a big part of adelic physics! Not surprisingly, the generalization of zeta function was proposed already by Dedekind (see `http://tinyurl.com/yarwbo6h`).

1. The definition of Dedekind zeta function ζ_K relies on the product representation and analytic continuation allows to deduce ζ_K elsewhere. One has a product over prime ideals of K/Q of rationals with the factors $1/(1-p^{-s})$ associated with the ordinary primes in Riemann zeta replaced with the factors $X(P) = 1/(1 - N_{K/Q}(P)^{-s})$, where P is prime for the integers $O(K)$ of extension and $N_{K/Q}(P)$ is the norm of P in the extension. In the region $s > 1$ where the product converges, ζ_K is non-vanishing and $s = 1$ is a pole of ζ_K. The functional identifies of ζ hold true for ζ_K as well. Riemann hypothesis is generalized for ζ_K.

2. It is possible to understand ζ_K in terms of a physical picture. By the results of `http://tinyurl.com/yckfjgpk` one has $N_{K/Q}(P) = p^r$, $r > 0$ integer. This implies that one can arrange in ζ_K all primes P for which the norm is power or given p in the same group. The prime ideals p of ordinary integers decompose to products of prime ideals P of the extension: one has $p = \prod_{r=1}^{g} P_r^{e_r}$, where e_r is so called ramification index. One can say that each factor of ζ decomposes to a product of factors associated with corresponding primes P with norm a power of p. In the language of physics, the particle state represented by p decomposes in an improved resolution to a product of many-particle states consisting of e_r particles in state P_r, very much like hadron decomposes to quarks.

   The norms of $N_{K/Q}(P_r) = p^{d_r}$ satisfy the condition $\sum_{r=1}^{g} d_r e_r = n$. Mathematician would say that the prime ideals of Q modulo p decompose in n-dimensional extension K to products of prime power ideals $P_r^{e_r}$ and that P_r corresponds to a finite field $G(p, d_r)$ with algebraic dimension d_r. The formula $\sum_{r=1}^{g} d_r e_r = n$ reflects the fact the dimension n of extension is same independent of p even when one has $g < n$ and ramification occurs.

   Physicist would say that the number of degrees of freedom is n and is preserved although one has only $g < n$ different particle types with e_r particles having d_r internal degrees of freedom. The factor replacing $1/(1 - p^{-s})$ for the general prime p is given by $\prod_{r=1}^{g} 1/(1 - p^{-e_r d_r s})$.

3. There are only finite number of ramified primes p having $e_r > 1$ for some r and they correspond to primes dividing the so called discriminant D of the irreducible polynomial P defining the extension. $D\ mod\ p$ obviously vanishes if D is divisible by p. For second order polynomials $P = x^2 + bx + c$ equals to the familiar $D = b^2 - 4c$ and in this case the two roots indeed co-incide. For quadratic extensions with $D = b^2 - 4c > 0$ the ramified primes divide D.

   Remark: Resultant $R(P, Q)$ and discriminant $D(P) = R(P, dP/dx)$ are elegant tools used by number theorists to study extensions of rationals defined by irreducible polynomials (see `http://tinyurl.com/oyumsnk` and `http://tinyurl.com/p67rdgb`). From Wikipedia articles one finds an elegant proof for the facts that $R(P,Q)$ is proportional to the product of differences of the roots of P and Q, and D to the product of squares for the differences of distinct roots. $R(P,Q)=0$ tells that two polynomials have a common root. $D \ mod \ p=0$ tells that polynomial and its derivative have a common root so that there is a degenerate root modulo p and the prime is indeed ramified. For modulo p reduction of P the vanishing of $D(P) \ mod \ p$ follows if D is divisible by p. There exists clearly only a finite number of primes of this kind.

Most primes are unramified and one has maximum number n of factors in the decomposition and $e_r=1$: maximum splitting of p occurs. The factor $1/(1-p^{-s})$ is replaced with its n:th power $1/(1-p^{-s})^n$. The geometric interpretation is that space-time sheet is replaced with n-fold covering and each sheet gives one factor in the power. It is also possible to have a situation in which no splitting occurs and one as $e_r=1$ for one prime $P_r=p$. The factor is in this case equal to $1/(1-p^{-ns})$.

From Wikipedia (see `http://tinyurl.com/yckfjgpk`) one learns that for Galois extensions L/K the ratio ζ_L/ζ_K is so called Artin L-function of the regular representation (group algebra) of Galois group factorizing in terms of irreps of $Gal(L/K)$ is *holomorphic* (no poles!) so that ζ_L must have also the zeros of ζ_K. This holds in the special case $K=Q$. Therefore extension of rationals can only bring new zeros but no new poles!

1. This result is quite far reaching if one accepts the hypothesis about super-symplectic conformal weights as zeros of ζ_K and the conjecture about coupling constant evolution. In the case of $\zeta_{F,K}$ this means new poles meaning new conformal weights due to increased complexity and a modification of the conjecture for the coupling constant evolution due to new primes in extension. The outcome looks physically sensible.

2. Quadratic field $Q(\sqrt{m})$ serves as example. Quite generally, the factorization of rational primes to the primes of extension corresponds to the factorization of the minimal polynomial for the generating element θ for the integers of extension and one has $p=P_i^{e_i}$, where e_i is ramification index. The norm of p factorizes to the produce of norms of $P_i^{e_i}$.

   Rational prime can either remain prime in which case x^2-m does not factorize mod p, split when x^2-m factorizes mod P, or ramify when it divides the discriminant of $x^2-m=4m$. From this it is clear that for unramfied primes the factors in ζ are replaced by either $1/(1-p^{-s})^2$ or $1/(1-p^{-2s})=1/(1-p^{-s})(1+p^{-s})$. For a finite number of unramified primes one can have something different.

   For Gaussian primes with $m=-1$ one has $e_r=1$ for $p \ mod \ 4=3$ and $e_r=2$ for $p= \ mod \ 4=1$. z_K therefore decomposes into two factors corresponding to primes $p \ mod \ 4=3$ and $p \ mod \ 4=1$. One can extract out Riemann zeta and the remaining factor

$$\prod_{p \ mod \ 4=3} \frac{1}{(1-p^{-s})} \times \prod_{p \ mod \ 4=1} \frac{1}{(1+p^{-s})}$$

   should be holomorphic and without poles but having possibly additional zeros at critical line. That ζ_K should possess also the poles of ζ as poles looks therefore highly non-trivial.

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