

Article

Does $M^8 - H$ Duality Reduce Classical TGD to Octonionic Algebraic Geometry? (Part I)

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Abstract

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8 - H$ duality is one of these approaches. The beauty of $M^8 - H$ duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.
2. It will be shown how $M^8 - H$ duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in M^8 would be algebraic surfaces identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o_c decomposing as $o_c = q_c^1 + q_c^2 I^4$ and projected to a Minkowskian sub-space M^8 of complexified O . Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm $q_c \bar{q}_c$ appearing in $RE(P)$ or $IM(P)$ caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for for quaternions at light-cone boundaries.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients can give rise to associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces

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for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and $M^8 - H$ correspondence could generalize.

- It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k , $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i . In the generic case this gives 3-D surface.

In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H , and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i . Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

- This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutativity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. Classical physics is an exact part of quantum physics in TGD and during years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

1.1 Various approaches to classical TGD

1.1.1 World of classical worlds

The first approach is based on the geometry of the "world of classical worlds" (WCW) [14, 12, 23].

1. The study of classical field equations led rather early to the realization that preferred extremals of twistor lift of Kähler action with Minkowskian signature of induced metric define a slicing of space-time surfaces defined by 2-D string world sheets and partonic two-surfaces locally orthogonal to them. The interpretation is in terms of position dependent light-like momentum vector and polarization vector defining the local decompositions $M^2(x) \times E^2(x)$ of tangent space integrating to a foliation by partonic 2-surfaces and string world sheets. I christened this structure Hamilton-Jacobi structure. Its Euclidian counterpart is complex structure in Euclidian regions of space-time surface.
2. The formulation of quantum TGD in terms of spinor fields in WCW [21] leads to the conclusion that WCW must have Kähler geometry [14, 12] and has it only if it has maximal group of isometries identified as symplectic transformations of $\delta M_{\pm}^4 \times CP_2$, where δM_{\pm}^4 denotes light cone boundary two which upper/lower boundary of causal diamond (CD) belongs. Symplectic Lie algebra extends naturally to supersymplectic algebra (SSA).

Space-time surfaces would be preferred extremals of twistor lift of Kähler action [29] and the conditions realizing strong form of holography (SH) would state that sub-algebra of SSA isomorphic with it and its commutator with SSA give rise to vanishing Noether charges and these charges annihilate physical states or create zero norm states from them.

1.1.2 Twistor lift of TGD

Second approach to preferred extremals is based on TGD version [20, 28, 27, 29] of twistor Grassmann approach [8, 10, 9].

1. The twistor lift of TGD leads to a proposal that space-time surfaces can be represented as sections in their 6-D twistor spaces identified as twistor bundles in the product $T(H) = T(M^4) \times T(CP_2)$ of 6-D twistor spaces of M^4 and CP_2 . Twistor structure would be induced from $T(H)$. Kähler action can be lifted to the level of twistor spaces only for $M^4 \times CP_2$ since only for these spaces twistor space allows Kähler structure [2]. Twistors were originally introduced by Penrose with the motivation that one could apply algebraic geometry in Minkowskian signature. The bundle property is extremely powerful and should be consistent with the algebraic geometrization at the level of M_c^8 . The challenge is to formulate the twistor lift at the level of M^8 .
2. The twistor lift of Kähler action contains also volume term. Field equations have two kinds of solutions. For the solutions of first kind the dynamics of volume term and Kähler action are coupled and the interpretation is in terms of interaction regions. Solutions of second kind are minimal surfaces and extremals of both Kähler action and volume term, whose dynamics decouple completely and all coupling constants disappear from the dynamics. These extremals are natural candidates for external particles. For these solutions at least the field equations reduce to the existence of Hamilton-Jacobi structure. The completely universal dynamics of these regions suggests interpretation in terms of maximal quantum criticality characterized by the extension of the usual conformal invariance to its quaternionic analog.
3. A connection with zero energy ontology (ZEO) emerges. Causal diamond (CD, intersection of future and past directed light-cones of M^4 with points replaced by CP_2) would naturally determine the interaction region to which external particles enter through its 2 future and past boundaries. But where does ZEO emerge?

1.1.3 $M^8 - H$ duality

The third approach is based on number theoretic vision [18, 19, 17, 24].

1. $M^8 - H$ duality [19, 24, 25] means that one can see space-times as 4-surfaces in either M^8 or $H = M^4 \times CP_2$. One could speak "number theoretical compactification" having however nothing to do with stringy version of compactification, which is dynamical. $M^8 - H$ duality suggests that space-time surfaces in $H = M^4 \times CP_2$ are images of space-time surfaces in M^8 or actually of M^8 projections of complexified space-time surfaces in M_c^8 identified as space of complexified octonions. These space-time surfaces could contain the integrated distributions of string world sheets and partonic 2-surfaces mentioned in the previous item. Space-time surfaces must have associative tangent or normal space for $M^8 - H$ correspondence to exist.
2. The fascinating possibility mentioned already earlier is that these surfaces could correspond to zero loci for real or imaginary parts of real analytic octonionic polynomials $P(o) = RE(P) + IM(P)I_4$, I_4 an octonionic imaginary unit orthogonal to quaternionic ones. The condition $IM(P) = 0$ ($RE(P) = 0$) would give associative (co-associative) space-time surface. In the simplest case these functions would be polynomials so that one would have algebraic geometry for algebraically 4-D complex surfaces in 8-D complex space.

Remark: The naive guess that space-time surfaces reduces to quaternionic curves in quaternionic plane fails due to the non-commutativity of quaternions meaning that one has $P(o) = P(q_1, q_2, \bar{q}_1, \bar{q}_2)$ rather than $P(o) = P(q_1, q_2)$.

3. One could also consider the possibility that the tangent spaces of space-time surfaces in H are associative or co-associative [24]. This is not necessary although it seems that this might be the case for the known extremals. If this holds true, one can construct further preferred extremals by functional composition by generalization of $M^8 - H$ correspondence to $H - H$ correspondence.
4. I have considered also the possibility of quaternion analyticity in the sense of generalization of Cauchy-Riemann equations, which tell that left- or right quaternionic differentiation makes sense [34]. It however seems that this approach is not promising. The conditions are quite too restrictive and bring nothing essentially new. Octonion/quaternion analyticity in the above mentioned sense does not require the analogs of Cauchy-Riemann conditions.

1.2 Could one identify space-time surfaces as zero loci for octonionic polynomials with real coefficients?

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.
2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by CP_2 points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.

3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [32]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials $P(o)$ containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with $RE(P) = 0$ can transform to $IM(P) = 0$ region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.
4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adèle for given extension of rationals. Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.
5. Also a connection with infinite primes is suggestive [19]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

1.3 Topics to be discussed

1.3.1 Key notions and ideas of algebraic geometry

Before going of octonionic algebraic geometry, I will discuss basic notions of algebraic geometry such as algebraic variety (see <http://tinyurl.com/hl6sjmz>), - surface (see <http://tinyurl.com/y8d5wsnj>), and - curve (see <http://tinyurl.com/nt6tkey>), rational point of variety central for TGD view about cognitive representation, elliptic curves (see <http://tinyurl.com/lovksny>) and - surfaces (see <http://tinyurl.com/yc33a6dg>), and rational points (see <http://tinyurl.com/ybbnysu>) and potentially rational varieties (see <http://tinyurl.com/yabl4xt>). Also the notion of Zariski topology (see <http://tinyurl.com/h5pv4vk>) and Kodaira dimension (see <http://tinyurl.com/yadoj2ut>) are discussed briefly. I am not a mathematician. What hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

1.3.2 $M^8 - H$ duality

$M^8 - H$ duality [25, 19, 24] would reduce classical TGD to the algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. Space-time surfaces in M^8 would be algebraic varieties identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o decomposing as $o = q_c^1 + q_c^2 I_4$ and projected to a Minkowskian sub-space M^8 of o . Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

The basic problem is to understand the map mediating $M^8 - H$ duality mapping the point (m, e) of $M^8 = M_0^4 \times E^4$ to a point (m, s) of $M_0^4 \times CP_2$, where M_0^4 point is obtained as a projection to a suitably chosen $M_0^4 \subset M^8$ and CP_2 point parameterizes the tangent space as quaternionic sub-space containing preferred $M_0^2(x) \subset M^4(x)$. This map involves slightly non-local information and could allow to understand why the preferred extremals at the level of H are determined by partial differential equations rather than algebraic equations. Also the generalization to the level of twistor lift is briefly touched.

1.3.3 Challenges of the octonionic algebraic geometry

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative/quaternionic surfaces as the zero loci of their imaginary/real parts in quaternionic sense. Here the intuition comes from the idea that the octonionic polynomials map from octonionic space O to second octonionic space W . Real and imaginary parts in W are quaternionic/co-quaternionic. These planes correspond to surfaces in O defined by the vanishing of real/imaginary parts, and the natural guess is that they are quaternionic/co-quaternionic, that is associative/co-associative.

This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction (see <http://tinyurl.com/ybuy1a2k>) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a $M^8 - H$ correspondence could generalize (maybe even TGD!).

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates z_i^k , $k = 1, 2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o = q_1 + q_2 I_4$, call this component X_i . In the generic case this gives 3-D surface.

In this generic case $M^8 - H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H , and only determines the boundary conditions of the dynamics in H determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i . Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [1] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in H in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutativity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

The easiest manner to kill the idea of $M^8 - H$ duality in the proposed form is to prove that the zero loci for imaginary/real parts of octonionic polynomials with real coefficients cannot be associative/co-associative. This would force to assume it holds true at the 3-D sub-varieties of 4-varieties: this option would be the minimal one. I hope that some professional mathematician would bother to check this.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

2 Some basic notions, ideas, results, and conjectures of algebraic geometry

In this section I will summarize very briefly the basic notions of algebraic geometry needed in the sequel.

2.1 Algebraic varieties, curves and surfaces

The basic notion of algebraic geometry is algebraic variety.

1. One considers affine space A^n with n coordinates x^1, \dots, x^n having values in a number field K usually assumed to be algebraically closed (note that affine space has no preferred origin like linear space). Algebraic variety is defined as a solution of one or more algebraic equations stating the vanishing of polynomials of n variables: $P^i(x^1, \dots, x^n) = 0, i = 1, \dots, r \leq n$. One can restrict the coefficients of polynomials to p -adic number field or its extension to an extension of rationals. One talks about polynomials on $k \subset K$.
2. The basic condition is that the variety is not a union of disjoint varieties. This for instance happens, when the polynomial $P(x^1, \dots, x^n)$ defining co-dimension 1 manifold is product of polynomials $P = \prod_r P_r$. Algebraic variety need not be a manifold meaning that it can have singular points. For instance, for co-dimension 1 variety the Jacobian matrix $\partial P / \partial x^i$ of the polynomial can vanish at singularity.
3. One can define projective varieties (see <http://tinyurl.com/ybsqvy3r>) in projective space P^n having coordinatization in terms of $n + 1$ homogenous coordinates (x^1, \dots, x^{n+1}) in K with points differing by an overall scaling identified. Projective variety is defined as zero locus of homogenous polynomials of $n + 1$ coordinates so that solutions remain solutions under the overall scaling of all coordinates. By identifying the points related by scaling one obtains a surface in P^n . Grassmannian of linear space V^n (not affine space!) is a projective spaces defined as space of k -planes of V^n . These spaces are encountered in twistor Grassmannian approach to scattering amplitudes.

For polynomials of single variable one obtains just the roots of $P_n(x) = 0$ in an algebraic extension assignable to the polynomial. For several variables one can in principle proceed step by step by solving variable x^1 as algebraic function of others from $P_1(x^1, \dots, x^n) = 0$, proceed to solve x^2 from $P_2(x^1(x^2, \dots), x^2, \dots) = 0$ as algebraic function of the remaining variables, and so on. The algebraic functions involved get increasingly complex but in some exceptional situations the solution has parametric representation in terms of *rational* rather than algebraic functions of parameters t^k . For co-dimension $d_c > 1$ case the intersection of surfaces $P^i = 0$ need not be complete and the tangent spaces of the hyper-surfaces $P^i = 0$ need not intersect transversally in the generic case. Therefore $d_c > 1$ case is not gained so much attention as $d_c = 1$ case.

A more advanced treatment relies on ring theory by assigning to polynomials a ring as the ring of polynomials in the space involved divided by the ring of polynomials vanishing at zero loci of polynomials P^i .

1. The notion of ideal is central and determined as a subring invariant under the multiplication by elements of ring. Prime ideal generalizes the notion of prime and one can say that the notion of integer generalizes to that of ideal. One can also define the notion of fractional ideal.
2. Zariski topology (see <http://tinyurl.com/h5pv4vk>) replacing the topology based on real norm is second highly advanced notion. The closed sets in this topology are algebraic varieties of various dimensions. Since the complement of any algebraic variety is open set this topology and open also in the ordinary real topology, this topology is considerable rougher than the ordinary than the ordinary topology.

Some remarks from the point of view of TGD are in order.

1. In the scenario inspired by $M^8 - H$ duality one has co-dimension 4 surfaces in 8-D complex space. Octonionicity of polynomials however implies huge symmetries since the polynomial is determined by single real polynomial of real variable, whose values at finite number of points determined the polynomial.
2. In TGD the extension of rationals can be assumed to contain also powers for some root of e since in p-adic context this gives rise to a finite-dimensional extensions due to the fact that e^p is ordinary p-adic number. Also a restriction to a finite field are possible and restriction of rational coefficients to their modulo p counterparts reduces the polynomial to polynomial in finite field. This reduction is used as a technical tool. In the case of Diophantine equations (see <http://tinyurl.com/nt6tkey> and <http://tinyurl.com/y8hm4zce>) the coefficients are restricted to be integers.
3. In adelic TGD [31] [36] the number fields involved are reals and extensions of p-adic numbers. The coefficient field for the coefficients of polynomials would be naturally extension of rationals or extension of p-adics induced by it. The coefficients of polynomials serve as coordinates of adelic WCW. p-Adic numbers are not algebraically closed and one must assume an extension of p-adic numbers from that for the coefficients one to allow maximal number of roots.

This suggests an evolutionary process [38] extending the number field for the coefficients of polynomials. Arbitrary root of polynomial for given extension can be realized only if the original extension is extended further. But this allows polynomial coefficients in this new extension: WCW is now larger. Now one has however roots in even larger extension so that the unavoidable outcome is number theoretic evolution as increase of complexity.

4. What is so remarkable is that octonionic polynomials with rational coefficients could be determined by their values at finite set of points for a polynomial of real argument once the order of polynomial is fixed. Real coordinate corresponds to preferred time axis naturally. A cognitive representation consisting of finite number of rational points could fix the entire space-time surface! This would extend ordinary holography to its discrete variant!

5. Algebraic variety is rather simple object as compared to the solutions of partial differential equations encountered in physics - say those for minimal surfaces. Now one must fix boundary values or initial values at $n - 1$ -dimensional surface to fix the solution. For integrable theories the situation can change. In TGD SH suggests that the classical solutions are determined by data at 2-surfaces, which together with conformal invariance could reduce the data to one-dimensional data specified by a polynomial. $M^8 - H$ correspondence allows to consider this option seriously.
6. $M^8 - H$ duality suggests that space-time surfaces are co-dimension $d_c = 4$ algebraic curves in M^8 . Could space-time surfaces define closed sets for the analog of Zariski topology? Could string world sheets and partonic 2-surfaces do the same inside space-time surfaces? An interesting question is whether this generalizes also to the level of imbedding space H and could perhaps define a topology rougher than real topology in better accord with the notion of finite measurement resolution.

2.2 About algebraic curves and surfaces

The realization $M^8 - H$ correspondence to be considered allows to understand space-time surfaces as 4-D complex algebraic surfaces X_c^4 in the space o of complexified octonions projected to real sub-space of O^c with Minkowskian signature. Due to the non-commutativity of quaternions, the reduction of space-time surfaces to curves in quaternionic plane is not possible. Despite this it is instructive to start from the algebraic geometry of curves and surfaces.

2.2.1 Degree and genus of the algebraic curve

Algebraic curve is defined as zero locus of a polynomial $P(x^1, x^2, \dots, x^n)$ with x^n in some - preferably algebraically closed - number field K and coefficients in some number field $k \subset K$. In adelic physics K corresponds to real or complex numbers and k to the extension of rationals defining adeles. In p-adic sectors k corresponds to the extension of p-adic numbers induced by k . In general roots belong to an extension of k .

Degree, genus, and Euler characteristic are the basic characterizers of algebraic curve.

1. The degree d of algebraic curve corresponds to the highest power for the variables appearing in the polynomial. One can also define multi-degree in an obvious manner. A useful geometric interpretation for the degree is that line intersects curve (also complex) of degree d in at most d points as is clear from the fact that the equation of curve reduces the equation for curve to an equation for the roots of d :th order polynomial of single variable.
2. Also the genus g of the curve (see <http://tinyurl.com/ybm3wfue>) is important characteristic. One can distinguish between topological genus, geometric genus and arithmetic genus. For curves these notions are equivalent. The connection between genus and degree d of non-singular algebraic curve is very useful:

$$g = \frac{(d-1)(d-2)}{2} . \tag{2.1}$$

Spherical topology for complex curves corresponds to $n = 1$ and $n = 2$.

A more general formula reads as:

$$g = \frac{(d-1)(d-2)}{2} + \frac{n_s}{2} . \tag{2.2}$$

Here n_s is the number of holes of the curve behaving like holes and increasing the genus.

3. Euler characteristic (for Euler characteristic see <http://tinyurl.com/pp52zd4>) is a homological invariant making sense in arbitrary dimension and also for manifolds. Homological definition based on simplicial homology relies on counting of simplexes of various dimension. The definition in terms of dimensions of homology groups H_n is given by

$$\chi = b_0 - b_1 + b_2 \dots + (-1)^n b_n \quad , \quad (2.3)$$

where b_k is the dimension of k :th homology group (see <http://tinyurl.com/j48ojys>).

The following gives the engineering rules for obtaining Euler characteristic of the surface obtained from simpler building blocks. Note that algebraic variety property is not essential here.

1. Euler characteristic is homotopy invariant so that it does not change one adds homologically trivial space such as E^n as a Cartesian factor.
2. χ is additive under disjoint union. Inclusion-exclusion principle states that if M and N intersect, one has $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$.
3. Euler characteristic for the connected sum $A \# B$ of n -dimensional manifolds obtained by drilling balls B^n from summands, giving opposite orientation to the boundaries of the hole, and connecting with cylinder $D \times S^{n-1}$ is given by $\chi(A) + \chi(B) - \chi(S^{n-1})$. One has $\chi(S^2) = 2$ and $\chi(D^2) = 1$.
4. The Euler characteristic for product $M \times N$ is $\chi(M) \times \chi(N)$.
5. The Euler characteristic for N -fold covering space M_n is $N \times \chi(M)$ with a correction term coming from the singularities of the covering (ramified covering space).
6. For a fibration $M \rightarrow B$ with fiber S , which differs from fiber bundle in that the fibers are only homeomorphic, one has $\chi(M) = \chi(B) \times \chi(S)$.

Euler characteristic and the genus of 2-surface (or complex) curve are related by the equation

$$\chi = 2(1 - g) \quad . \quad (2.4)$$

having values 2, 0, -2, If the 2-surface has n_s holes (punctures), one has

$$\chi = 2(1 - g) - n_s \quad . \quad (2.5)$$

Punctures must be distinguished from singularities at which some sheets of covering meet at single point.

A formal generalization of the definition of genus for varieties in terms of Euler characteristic makes sense.

$$g = -\frac{\chi}{2} + 1 - \frac{n_s}{2} \quad . \quad (2.6)$$

Disk has genus 1/2 and drilling of n holes increases genus by $n/2$. Pair of holes gives same contribution to g and the cylinder connecting the holes. Note that for complex curves the definition of puncture is obvious. For real curves the puncture would mean missing point of the curve.

The latter definitions of genus can be identified in terms of Euler characteristic also for higher-dimensional varieties. For curves these notions are equivalent if there are no singularities. For algebraic curves g is same for the real and complex variants of the curve in RP_1 and CP_1 respectively.

2.2.2 Elliptic curves and elliptic surfaces

Elliptic curves (see <http://tinyurl.com/lovksny>) are cubic curves with no singularities (cusps or self-intersections) having representation of form $y^2 - x^3 - ax - b = 0$. These singularities can occur only at special values of parameters ($(a = 0, b = 0)$). Since the degree equals to $d = 3$, elliptic curve has genus $g = 1$.

Elliptic curves allow a group of Abelian symmetries generated by a finite number of generators. The emergence of abelian group structure can be intuitively understood as follows.

1. Given line intersects the curve of degree 3 in at most 3 points. Let P and Q be two of these points. Then there can be also a third intersection point R and by the Z^2 symmetry changing the sign of y also the reflection of R - identify it as $-R$ - belongs to the curve. Define the sum of $P + Q$ to be $-R$.

The actual proof is slightly more complicated since the number of intersection points for the line with curve can be also 2 or 1. By writing explicit expressions for the coordinates x_R and y_R , one can also find that they are indeed rational if the points P and Q are rational. If the elliptic curve as single rational point it has infinite number of them.

2. The generators with finite order give rise to torsion. The rank of generators of infinite order is called rank and conjectured to be arbitrarily large (see <http://tinyurl.com/lovksny>). Therefore elliptic curve is an Abelian group and one talks about Abelian variety. If elliptic curve contains a rational point it contains entire lattice of rational points obtained as shifts of this point.

Remark: Complex elliptic curves are 2-surfaces in complex projective plane CP_2 and therefore highly interesting from TGD point of view. $g = 1$ partonic 2-surfaces would in TGD framework correspond to second generation fermions [11]. Abelian varieties define a generalization of elliptic curves to higher dimensions and simplest space-time surfaces allowing also large cognitive representations could correspond to such.

Elliptic surfaces (see <http://tinyurl.com/yc33a6dg>) are fibrations with an algebraic curve as base space and elliptic curve as fiber (fibration is more general notion than fiber space since the fibers are only homeomorphic). The singular fibers failing to be elliptic curves have been classified by Kodaira.

2.3 The notion of rational point and its generalization

The notion of algebraic integer (see <http://tinyurl.com/y8z389a7>) makes sense for any number field as a root of a monic polynomial (polynomial with integer coefficients with coefficient of highest power equal to unity). The field of fractions for given number field consists of ratios of algebraic integers. The same is true for the notion of prime. The more precise definition forces to replace integers and primes with ideals.

Rational varieties are expressible as maps defined by rational functions with rational coefficients in some extension of Q and contain infinite number of rational points. If the variety is not rational, one can ask whether it could allow a dense set of rational points with rational number replaced with the ratio of algebraic integers for some extension of Q . This leads to the idea of potentially rational point, and one can classify algebraic varieties according to whether they are potentially rational or not. The variety is potentially rational if it allows a parametric representation using rational functions. Otherwise the parametric representation involves algebraic functions such as roots of rational functions.

The interpretation in terms of cognition would be that large enough extension makes the situation "cognitively easy" since cognitive representations involving fermions at the rational points and defining discretizations of the algebraic variety could be arbitrary large. The simpler the surface is cognitively, the larger the number of rational points or potentially rational points is.

Complexity of algebraic varieties is measured by Kodaira dimension d_K (see <http://tinyurl.com/yadoj2ut>). The spectrum for this dimension varies in the range $(-\infty, 0, 1, 2, \dots, d)$, where d is the algebraic

dimension of the variety. Maximal value equals to the ordinary topological dimension d and corresponds to maximal complexity: in this case the set of rational points is finite. Minimal Kodaira dimension is $d_K = -\infty$: in this case the set of rational points is infinite. Rational surfaces are maximally simple and this corresponds to the existence of parametric representations using only rational functions.

2.3.1 Rational points for algebraic curves

The sets of rational points for algebraic curves are well understood. Mordelli conjecture proved by Falting as a theorem (see <http://tinyurl.com/y9oq37ce>) states that a curve over Q with genus $g = (d - 1)(d - 2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

1. Sphere CP_1 in CP_2 has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [3, 5].

$g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in CP_2 with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve $y^2 - x^3 - ax - b$ in CP_2 (see <http://tinyurl.com/lovksny>) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

$g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and CP_2 as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d - 1)(d - 2)/2$ in CP_2 (one has $g = 0, 0, 2, 3, 6, 10, \dots$). For $d > 2$, in particular $d = 3$, there are no rational points.

3. $g \geq 2$ curves do not allow a dense set of rational points nor even potentially cense set of rational points.

Remark: In TGD framework algebraic varieties could be zero loci of octonionic polynomials and have algebraic dimension 4 so that the classification for algebraic curves does not help. Octonion analyticity must bring in symmetries which simplify the situation.

2.3.2 Enriques-Kodaira classification

The tables of (see <http://tinyurl.com/ydelr4np>) give an overall view about the Enriques-Kodaira classification of algebraic curves, surfaces, and varieties in terms of Kodaira dimension (see <http://tinyurl.com/yadoj2ut>).

1. For instance, general curves ($g \geq 2$) have $d_K = 1$, elliptic curves ($g = 1$) have $d_K = 0$ and projective line ($g = 0$) has $d_K = -\infty$. $CP_1 \subset CP_2$ is a rational curve so that rational points are dense. Elliptic curves allow infinite number or rational points forming an Abelian group if they containing single rational point and are therefore cognitively easy.
2. Algebraic varieties (with real dimension $d_R = 4$ in complex case) with $d_K = 2$ are surfaces of general type, elliptic surfaces (see <http://tinyurl.com/yc33a6dg>) have $d_K = 1$, surfaces with attribute abelian, hyper-elliptic, K3, and Enriques, have $d_K = 0$.

Remark: All real 2-surfaces are hyper-elliptic for $g \leq 2$, in other words allow Z_2 as global conformal symmetry. Genus-generation correspondence [11] for fermions allows to assign to the 3 lowest generations of fermions hyper-elliptic partonic 2-surfaces with genus $g = 0, 1, 2$. These surfaces

would have $d_K = 0$ and be rather simple as real surfaces in Kodaira classification. Could one regard them as M^4 projection of complex hyper-elliptic surfaces of real dimension $d_R = 4$? $d_K = -\infty$ holds true for rational surfaces and ruled surfaces, which allow straight line through any point are maximally simple. In complex case the line would be CP_1 .

3. The Wikipedia article gives also a table about the classification of algebraic 3-folds. Real algebraic 3-surfaces might well occur in TGD framework. The twistor space for space-time surface might allow realization as complex 3-fold and since it has S^2 as fiber, it would naturally correspond to an uni-ruled surface with $d_K = -\infty$. The table shows that one can build higher dimensional algebraic varieties with $d_K < d$ from lower-dimensional ones as fiber-space like structures, which based on fiber having $d_K < d$. 3-D Abelian varieties and Calabi-Yau 3-folds are complex manifolds with $d_K = 0$, which cannot be engineered in this manner.
4. Space-time surfaces would be surfaces of algebraic dimension 4. Wikipedia tables do not give direct information about this case but one can make guesses on basis of the three tables. Octonionic polynomials are analytic continuations of real polynomials of real variable, which must mean a huge simplification, which also favor cognitive representability. The best that one might have infinite sets of rational points. The examples about extremals of Kähler action does not however favor this wish.

Bombieri-Lang conjecture (see <http://tinyurl.com/y887yn5b>) states that, for any variety X of general type over a number field k , the set of k -rational points of X fails to be Zariski dense (see <http://tinyurl.com/jm9fh74>) in X . This means that , the k -rational points are contained in a *finite* union of lower-dimensional sub-varieties of X . In dimension 1, this is exactly Faltings theorem, since a curve is of general type if and only if it has $g \geq 2$. The conjecture of Vojta (see <http://tinyurl.com/y9sttuu4>) states that varieties of general type cannot be potentially dense. As will be found, these conjectures might be highly relevant for TGD.

3 Does $M^8 - H$ duality allow to use the machinery of algebraic geometry?

The machinery of algebraic geometry is extremely powerful. In particular, the number theoretical universality of algebraic geometry implies that same equations make sense for all number fields: this is just what adelic physics [31] [36] demands. Therefore it would be extremely nice if one could somehow use this machinery also in TGD framework as it is used in string models. How this could be achieved? There are several guide lines.

1. Twistor lift of TGD [20, 28, 27, 29] is now a rather well-established idea although a lot of work remains to be done with the details. Twistors were originally introduced in order to be able to use this machinery and involves complexification of Minkowski space M^4 to M_c^4 as an auxiliary tool. Complexification in sufficiently general sense seems to be a necessary auxiliary tool but it cannot be a trick (like Wick rotation) but something fundamental and here complexification at the level of M^8 is suggestive. In the sequel I will use M^4 for M_c^4 and M^8 for M_c^8 unless it is necessary to emphasize that M_c^8 is in question. The essential point is that the Euclidian metric is complexified and it reduces to a real metric in various sub-spaces defining besides Euclidian space also Minkowski spaces with varying signature when the complex coordinates are real or imaginary.
2. If $M^8 - H$ duality holds true, one can solve field equations in $M^8 = M^4 \times E^8$ by assuming that either the tangent space or normal space of the space-time surface X^4 is associative (quaternionic) at each point and contains preferred M^2 in its tangent space. M^2 could depend on x but $M^2(x)$ should integrate to a 2-surface. This allows to map space-time surface M^8 to a surface in $M^4 \times CP_2$

since tangent spaces are parameterized by points of CP_2 and CP_2 takes the role of moduli space. The image of tangent space as point of CP_2 is same irrespective of whether one has quaternions or complexified quaternions (H_c).

It came a surprise that associativity/co-associativity is possible only if the space-time surface is critical in the sense that some gradients of 8 complex components of the octonionic polynomial P vanish without posing them as additional conditions reducing thus the dimension of the space-time surface. This occurs when the coefficients of P satisfy additional conditions. One obtains associative/co-associative space-time regions and regions without either property and they correspond nicely to two solution types for the twistor lift of Kähler action.

3. Contrary to the original expectations, $M^4 \subset M_c^8$ must be identified as co-associative (co-quaternionic) subspace so that E^4 is the associative/quaternionic sub-space. This allows to have light-cone boundary as the counterpart of point-like singularity in ordinary algebraic geometry and also allows to understand the emergence of CDs and ZEO.

Remark: A useful convention to be used in the sequel. $RE(o)$ and $IM(o)$ denote the real and imaginary parts of the octonion in the decomposition $o = RE(o) + IM(o)I_4$ and $Re(o)$ and $Im(o)$ its real number valued and purely imaginary parts in the usual decomposition.

The problems related to the signature of M^4 have been a longstanding head-ache of M^8 duality.

1. The intuitive vision has been that the problems can be solved by replacing M^8 with its complexification M_c^8 identifiable as complexified octonions o . This requires introduction of imaginary unit i commuting with the octonionic units $E^k \leftrightarrow (1, I_1, \dots, I_7)$. The real octonionic components are thus replaced with ordinary complex numbers $z_i = x_i + iy_i$.
2. Importantly, complex conjugation $o \rightarrow \bar{o}$ changes only the sign of I_i but *not!* that of i so that the octonionic inner product $(o_1, o_2) = o_1 \bar{o}_2 = o_1^k o_2^l \delta_{k,l}$ becomes complex valued. Norm is equal to $\overline{O}O = \sum_i z_i^2$. Both norm and inner product are in general complex valued and real valued only in sub-spaces in which octonionic coordinates are real or imaginary. Sub-spaces have all possible signatures of metric. These sub-spaces are not closed under multiplication and this has been an obstacle in the earlier attempts based on the notion of octonion analyticity. This argument applies also to quaternions and one obtains signatures $(1, 1, 1, 1)$, $(1, 1, 1, -1)$, $(1, 1, -1, -1)$, and $(1, -1, -1, -1)$. Why just the usual Minkowskian signature $(1, -1, -1, -1)$ is physical, should be understood.

The convention consistent with that used in TGD corresponds to a negative length squared for space-like vectors and positive for time-like vectors. This gives $m = (o^0, io^1, \dots, io^7)$ with real o^k . The projection $M_c^8 \rightarrow M^8$ defines the projection of $X_c^4 \subset M_c^8$ to $X^4 \subset M^8$ serving as the pre-image of $X^4 \subset M^8$ in $M^8 - H$ correspondence.

3. o is not field anymore as is clear from the fact that $1/o = \bar{o}/o\bar{o}$ is formally infinite in Minkowskian sub-spaces, when octonion defines a light-like vector. o (and H_c) remains however a ring so that sum and products are well-defined but division can lead to problems unless one stays inside 7+7-dimensional light-cone with $Re(o\bar{o}) > 0$ ($Re(q\bar{q}) > 0$).

Although the number field structure is lost, one can still define polynomials needed to define algebraic varieties by requiring their simultaneous vanishing and rational functions make sense inside the light-cone. Also rational functions can be defined but poles are replaced with light-cones in Minkowskian section. Algebraic geometry would thus be forced by the complexification of octonions. This looks to me highly non-trivial! The extension of zeros and poles to light-cones making propagation possible could be a good reason for why Minkowskian signature is physical. Interestingly, the allowed octonionic momenta are light-like quaternions [29].

4. An interesting question is whether ZEO and the emergence of CDs relates to the failure of field property. It seems now clear that CDs must be assigned even with elementary particles and they could form an analog for the covering of manifold by coordinate patches (in TGD inspired theory of consciousness CDs would be correlates for perceptive fields for conscious entities assignable to CDs [38]). These observations encourage to ask whether the tips of CD should correspond to a pair formed by two poles/two zeros or by pole and zero assignable to positive and negative energy states. It turns out that the space-time surfaces in the interior of CD would naturally correspond to non-associative surfaces and only their 3-D boundaries would have associative 4-D tangent spaces allowing mapping to H by M^8 -duality, which is enough by holography.
5. The relationship between light-like 3-surface bounding Minkowskian and Euclidian space-time regions and light-like boundaries of CDs is interesting. Could also the partonic orbits be understood a singularities of octonionic polynomials with $IM(P) = RE(P) = 0$?

3.1 What does one really mean with $M^8 - H$ duality?

The original proposal was that M^8 duality should map the associative tangent/normal planes of associative/co-associative space-time surface containing preferred M^2 , call it M_0^2 , to CP_2 : the map read as $(m, e) \in M^4 \times E^4 \rightarrow (m, s) \in M^4 \times CP_2$. Eventually it became clear that the choice of M^2 can depend on position with $M^2(x)$ forming an integrable distribution to CP_2 : this would define what I have called Hamilton-Jacobi structures [25]. String like objects have minimal surface as M^4 projection for almost any general coordinate invariant action, and internal consistency requires that $M^2(x)$ integrate to a minimal surface. The details are however not understood well enough.

1. M^4 coordinate would correspond simply to projection to a fixed M_0^4 in the decomposition $M^8 = M_0^4 \times E_0^4$. One can however challenge this interpretation. How M_0^4 is chosen? Is it possible to choose it uniquely? Could also M^4 coordinates represent moduli analogous to CP_2 coordinates? What about ZEO?

There is an elegant general manner to formulate the choice of M_0^4 at the level of M^8 . The complexified quaternionic sub-spaces of M_c^8 (M^8) are parameterized by moduli space defining the quaternionic moduli. The moduli space in question is CP_2 . The choice of M_0^4 corresponds to fixing of the quaternionic moduli by fixing a point of CP_2 .

Warning: Note that one should be very careful in distinguishing between quaternionic as sub-spaces of M^8 and as the tangent space M^8 of given point of M^8 .

2. One can ask whether there could be a connection with ZEO, where CDs play a key role. Indeed, the complexified Minkowski inner product means that the complexified octonions (quaternions) inside M_c^8 (M_c^4) have inverse only inside 7-D (4-D) complexified light-cone and this would motivate the restriction of space-time surfaces inside future or past light-cone or both but not yet force CD.

If one allows rational functions and even meromorphic functions of octonionic or quaternionic variable, one could consider the possibility of restricting the space-time surface defined as their zeros to a maximally sized region containing no poles.

3. Consider complexified quaternions H_c . Poles $(q\bar{q})^{-n}$, $n \geq 1$ would correspond M^4 light-cone boundaries since $q\bar{q} = 0$ at them. Also zeros $q\bar{q} = 0$, for $n \geq 1$ correspond to light-like boundaries. Could one have two poles with with time-like distance defining CD or a pair of pole and zero?

There is also a possible connection with the notion of infinite primes [17]. The notion of infinite prime leads to the proposal that rationals defined as ratios of infinite integers but having unit real norm (and also p-adic norms) could correspond pairs of positive and negative energy states with identical total quantum numbers and located at opposite boundaries of CD. Infinite rationals can

be mapped to rational functions. Could positive energy states correspond to the numerators with zeros at second boundary of CD and negative energy states to denominators with zeros at opposite boundary of CD?

3.1.1 Is the choice of the pair (M_0^2, M_0^4) consistent with the properties of known extremals in H

It should be made clear that the notion of associativity/co-associativity (quaternionicity/co-quaternionicity) of the tangent/normal space need not make sense at the level of H . I shall however study this working hypothesis in the sequel.

The choice of the pair (M_0^2, M_0^4) means choosing preferred co-commutative (commutative) sub-space M_0^2 of M^8 defining a subspace of fixed co-quaternionic (quaternionic) sub-space $M_0^4 \subset M^8$.

Remark: M^4 should indeed be the co-associative/co-quaternionic subspace of M^8 if the argument about emergence of CDs is accepted and if $M^8 - H$ correspondence maps associative to associative and co-associative to co-associative.

M_0^4 in turn contains preferred M_0^2 defining co-commutative (hyper-complex) structure. Both M_0^2 and M_0^4 are needed in order to label the choice by CP_2 point (that is as a point of Grassmannian).

Is the projection to a fixed factor $M_0^4 \subset M_0^4 \times E^4$ as a choice of co-quaternionic moduli consistent with what we know about the extremals of twistor lift of Kähler action in H ? How could one fix M_0^4 from the data about the extremal in H ? One can make similar equations about the choice of M_0^2 as a fixed co-complex moduli characterized by a unit quaternion. Note that this choice is expected to relate closely to the twistor structure and Kähler structure.

It is best to check the proposal for the known extremals in H [25]. Consider first CP_2 type extremals for which M^4 projection is a piece of light-like geodesic.

1. The CP_2 projection for the image of $X^4 \subset M^8$ differs from single point only if the tangent space isomorphic to M^4 and parameterized by CP_2 point varies. Consider CP_2 type extremals for the twistor lift of Kähler action [32] in H having light-like geodesic as M^4 projection as an example. The light-like geodesic defines a light-like vector in the tangent space of CP_2 type extremal. This light-like vector together with its dual spans fixed M^2 , which however does not belong to the tangent space so that associative surface would not be in question.

What about co-associativity or associativity (the latter is favored by above argument)? This property should hold true for the pre-image of CP_2 type extremal in M^8 but I am not able to say anything about this. It is questionable to require this property at the level H but one can of course look what it would give.

What about associativity for CP_2 tangent space? The normal space of CP_2 type extremal is 3-D (!) since the only the light-like tangent vector of the geodesic and 2 vectors orthogonal to it are orthogonal to CP_2 tangent vectors. For Euclidian signature this would mean that tangent space is 5-D and cannot be associative but now the tangent space is 4-D. Can one still say that tangent space is associative. The co-associativity of the tangent space makes sense trivially. Can one conclude that CP_2 is co-associative.

The associativity for CP_2 tangent space might make sense since the tangent space is 4-D. The light-like vector k defines M_0^2 . The associativity conditions involving two tangent space vectors of CP_2 and the light-like vector k contracted with the corresponding octonion components. The contributions from the components of k to the associator should cancel each other. Since one can change the relative sign of the components of k , this mechanism does not seem to work for all components. Hence associativity cannot hold true. Neither does M_0^2 belong to the normal space since k and its dual are not orthogonal.

Could one conclude that CP_2 type extremal is co-associative in accordance with the original belief thanks to the light-like projection to M^4 ? This does not conform with what the singularity consid-

erations for the octonionic polynomials would suggest. Or is it simply not correct to try to apply associativity at the level of H . Or does $M^8 - H$ correspondence map associative tangent spaces to co-associative ones?

2. The normal space M^4 of CP_2 type extremals have all orientations characterized by its CP_2 projection. The normal space must contain the M_0^2 determined by the tangent of the light-like geodesic and this is indeed the case. Note that CP_2 type extremals cannot have entire CP_2 as CP_2 projection: they necessarily have hole at either end, which would be naturally be at the boundary of CD.

CP_2 type extremals seem to be consistent with $M^8 - H$ correspondence. It however seems that one cannot fix the choice of M_0^4 uniquely in terms of the properties of the extremal. There is a moduli space for M_0^4 :s defined by CP_2 and obviously codes for moduli for quaternion structures in octonionic space. The distributions of $M^2(x)$ (minimal surfaces) would code for quaternion structures (decomposition of octonionic coordinates to quaternionic coordinates in turn decomposing to pairs of complex coordinates).

Consider next the associativity condition for cosmic strings in $X^2 \times Y^2 \subset M^4 \times CP_2$. Now CP_2 projection is 2-D complex surfaces and M^4 projection is minimal surface. Situation is clearly associative. How unique the choice of M_0^4 is now?

1. Now $M^2(x)$ depends on position but $M^2(x)$:s define an integrable distribution defining string orbit X^2 as a minimal surface. M_0^4 must contain all surfaces $M^2(x)$, which would fix M_0^4 to a high degree for complex enough cosmic strings.
2. Each point of X^2 corresponds to the same partonic surface $Y^2 \subset CP_2$ labelling the tangent spaces for its pre-image in M^8 . All the tangent surfaces $M^2(x) \times E^2(y)$ for $X^2 \times Y^2 \subset M^8$ share only $M^2(x) \subset M_0^4$. M_0^4 must contain all tangent spaces $M^2(x)$ and the inverse image of $Y^2 \subset CP_2$ must belong to the orthogonal complement E^4 of M_0^4 . This is completely analogous with the condition $X^2 = X^2 \times Y^2 \subset M^4 \times CP_2$.

Consider a decomposition $M^8 = M_0^4 \times E^4$, $M_0^4 = M_0^2 \times E_0^2$. If the inverse image of Y^2 at point x belongs to E^4 , the M_0^4 projection belongs to M_0^4 also in M^8 . If this does not pose any condition on the tangent spaces assignable to the points of Y^2 defining points of CP_2 , there are no problems. What could happen that the tangent spaces assignable to Y^2 could force the projection of the inverse image of Y^2 to intersect M_0^4 .

One should also understand massless extremals (MEs). How to choose M_0^4 in this case?

1. MEs are given as zeros of arbitrary functions of CP_2 coordinates and 2 M^4 coordinates u and v representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant and define $M_0^4 = M_0^2 \times E_0^2$ decomposition everywhere so that M_0^4 is uniquely defined. Same applies also when the directions are not constant. In the general case light-like direction would define the local tangent plane of string world sheet and local polarization plane. Since the dimension of M^4 projection is 4 there seems to be no problems involved.
2. Tangent plane of X^4 is parameterized by CP_2 coordinates depending on 2 coordinates u and v . The surface $X^4 \subset M^8$ must be graph for a map $M_0^4 \rightarrow E^4$ so that a 2-parameter deformation of M_0^4 as tangent plane is in question. The distribution of tangent planes of $X^4 \subset M^8$ is 2-D as is also the CP_2 projection in H .

To sum up, the original vision about the associativity properties of the known extremals at level of H survives. On the other hand, CDs emerge if M^4 corresponds to the co-associative part of O . Does this mean that $M^8 - H$ correspondence maps associative to co-associative by multiplying the quaternionic tangent space in M^8 by I_4 to get that in H and vice versa or that the notions of associative and co-associative do not make sense at the level of H ? This does not affect the correspondence since the same CP_2 point parametrizes both associative tangent space and its complement.

3.1.2 Space-time surfaces as co-dimension 4 algebraic varieties defined by the vanishing of real or imaginary part of octonionic polynomial?

If the theory intended to be a theory of everything, the solution ansatz for the field equations defining space-time surfaces should be ambitious enough: nothing less than a general solution of field equations should be in question.

1. One cannot exclude the possibility that all analytic functions of complexified octonionic variable with real Taylor or even Laurent coefficients. These would form a commutative and associative algebra. Space-time surfaces would be identified as their zero loci. This option is however number theoretically attractive and can also lead to problems with adelic physics. Since Taylor series at rational point need not anymore give a rational value.
2. Polynomials of complexified octonion variable o with real coefficients define the simplest option but also rational functions formed as ratios of this kind of polynomials must be considered. Polynomials form a non-associative ring allowing sum, product, and functional decomposition as basic operations. If the coefficients o_n of polynomials are complex numbers $o_n = a_n + ib_n$, a_n, b_n real, where i refers to the commutative imaginary unit complexifying the octonions, the ring is associative. It is essential to allow only powers o^n (or $(o - o_0)^n$ with $o_0 = a_0 + ib_0$, a_0, b_0 real numbers). Physically this means that a preferred time axis is fixed. This time axis could connect the tips of CD in ZEO.

One can write

$$P(o) = \sum_k p_k o^k \equiv RE(P)(q_1, q_2, \bar{q}_1, \bar{q}_2) + IM(P)(q_1, q_2, \bar{q}_1, \bar{q}_2) \times I_4, \quad p_k \text{ real}, \quad (3.1)$$

where the notations

$$o = q_1 + q_2 I_4, \quad q_i = z_i^1 + z_i^2 I_2, \quad \bar{q}_i = z_i^1 - z_i^2 I_2, \quad z_i^j = x_i^j + iy_i^j \quad (3.2)$$

Note that the conjugation does *not* change the sign of i . Due to the non-commutativity of octonions P^i as functions of quaternions are in general *not* analytic in the sense that they would be polynomials of q_i with real coefficients! They are however analytic functions of z_i . The real and imaginary parts of x_i^j correspond to Minkowskian and Euclidian signatures.

In adelic physics coefficients o_n of the octonionic polynomials define WCW coordinates and should be rational numbers or rationals in the extension of rationals defining the adèle. The polynomials form an associative algebra since associativity holds for powers o^n multiplied by real number. Thus complex analyticity crucial in algebraic geometry would be a key element of adelic physics.

3. If the preferred extremals correspond to the associative algebra formed by these polynomials, one could construct a completely general solution of the field equations as zero loci of their real or imaginary parts and build up of new solutions using algebra operation sum, product, and functional decomposition. One could identify space-time regions as associative or co-associative algebraic varieties in terms of these polynomials and they would form an algebra.

The motivation for this dream comes from 2-D electrostatics, where conducting surfaces correspond to curves at which the real part u or imaginary part v of analytic function $w = f(z) = u + iv$ vanishes. In electrostatics curves form families with curves orthogonal to each other locally and the map $w = u + iv \rightarrow v - iu$ defines a duality in which curves of constant potential and the curves defining their normal vectors are mapped to each other.

1. The generalization to the recent situation would be vanishing of the imaginary part $IM(P)$ or real part $RE(P)$ of the octonionic polynomial, where real and imaginary parts are defined via $o = q_c^1 + q_c^2 I_4$. One can consider also the possibility that imaginary or real part has constant value c are restricted to be rational so that one can regard the constant value set also as zero set for a polynomial with constant shift. Note that the rationals could be also complexified by addition of i . One would have

$$RE(P)(z_i^k) \quad \text{or} \quad IM(P)(z_i^k) = c \quad , \quad c = c_0 \text{ rational} \quad . \tag{3.3}$$

c_0 must be real. These two options should correspond to the situations in which tangent space or normal space is associative (associativity/co-associativity). Complexified space-time surfaces X_c^4 corresponding to different constant values c of imaginary or real part (with respect to i) would define foliations of M_c^8 by locally orthogonal 4-dimensional surfaces in M_c^8 such that normal space for surface X_c^4 would be tangent space for its co-surface.

2. It must be noticed that one has moduli space for the quaternionic structures even when M_0^4 is fixed. The simplest choices of complexified quaternionic space $H_c = M_{c,0}^4$ containing preferred complex plane $M_{c,0}^2$ and its orthogonal complement are parameterized by CP_2 . More complex choices are characterized by the choice of distribution of $M^2(x)$ integrable to (presumably minimal) 2-surface in M^4 . Also the choice of the origin matters as found and one has preferred coordinates. Also the 8-D Lorentz boosts give rise to further quaternionic moduli. The physically interesting question concerns the interpretation of space-time surfaces with different moduli. For instance, under which conditions they can interact?

The proposal has several extremely nice features.

1. Single real valued polynomial of real coordinate extended to octonionic polynomial and fixed by real coefficients in extension of rationals would determine space-time surfaces.
2. The notion of analyticity needed in concrete equations is just the ordinary complex analyticity forced by the octonionic complexification: there is no need for the application to have left- or right quaternion analyticity since quaternionic derivatives are not needed. Algebraically one has the most obvious guess for the counterpart of real analyticity for polynomials generalized to octonionic framework and there is no need for the quaternionic generalization of Cauchy-Riemann equations [7, 4] [7, 4] (<http://tinyurl.com/yb8134b5>) plagued by the problems with the definition of differentiation in non-commutative and non-associative context. There would be no problems with non-associativity and non-commutativity thanks to commutativity of complex coordinates with octonionic units.
3. The vanishing of the real or imaginary part gives rise to 4 conditions for 8 complex coordinates z_1^k and z_2^k allowing to solve z_2^k as algebraic functions $z_2^k = f^k(z_1^l)$ or vice versa. The conditions would reduce to algebraic geometry in complex co-dimension $d_c = 4$ and all methods and concepts of algebraic geometry can be used! Algebraic geometry would become part of TGD as it is part of M-theory too.

3.2 Is the associativity of tangent-/normal spaces really achieved?

The non-trivial challenge is to prove that the tangent/normal spaces are indeed associative for the two options. The surfaces X_c^4 are indeed associative/co-associative if one considers the *internal* geometry since points are in M_c^4 or its orthogonal complement.

One should however prove that X_c^4 are also associative *as sub-manifolds* of O and therefore have quaternionic tangent space or normal space at each point parameterized by a point of CP_2 in the case that tangent space containing position dependent M_c^2 , which integrate to what might be called a 2-D complexified string world sheet inside M_c^4 .

1. The first thing to notice that associativity and quaternionicity need not be identical concepts. Any surface with complex dimension $d < 4$ in O is associative and any surface with dimension $d > 4$ co-associative. Quaternionic and co-quaternionic surfaces are 4-D by definition. One can of course ask whether one should consider a generalization of brane hierarchy of M-theory also in TGD context and allow associativity in its most general sense. In fact, the study of singularity of o^2 shows that 6 and 5-dimensional surfaces are allowed for which the only interpretation would be as co-associative spaces. This exceptional situation is due to the additional symmetries increasing the dimension of the zero locus.
2. One has clearly quaternionicity at the level of o obtained by putting $Y = 0$ and at the level of the tangent space for the resulting surface. The tangent space should be quaternionic. The Jacobian of the map defined by P is such that it takes fixed quaternionic subspace $H_c \rightarrow M_{0,c}^4$ of O to a quaternionic tangent space of X^4 . The Jacobian applied to the vectors of H_c gives the octonionic tangent vectors and they should span a quaternionic sub-space.
3. The notion of quaternionic surface is rigorous. $M^8 - H$ correspondence could be actually interpreted in terms of the construction of quaternionic surface in M^8 . One has 4-D integrable distribution of quaternionic planes in O with given quaternion structure labelled by points of CP_2 and has representation at the level of H as space-time surface and should be preferred extremals. These quaternion planes should integrate to a slicing by 4-surfaces and their duals. One obtains this slicing by fixing the values 4 of the suitably defined octonionic coordinates P^i , $i = 1, \dots, 8$, to a real constants depending on the surface of the slicing. This gives a space-time surfaces for which tangent space-spaces or normal spaces are quaternionic.

The first guess for these coordinates P^i be as real or imaginary parts of real polynomials $P(o)$. But how to prove and understand this?

Could the following argument be more than wishful thinking?

1. In complex case an analytic function $w(z) = u + iv$ of $z = x + iy$ mediates a map between complex planes Z and W . One can interpret the imaginary unit appearing in w locally as a tangent vector along $u = \text{constant}$ coordinate line.
2. One can interpret also octonionic polynomials with real coefficients as mediating a map from octonionic plane O to second octonionic plane, call it W . The decomposition $P = P^1 + P^2)I_4$ would have interpretation in terms of coordinates of W with coordinate lines representing quaternions and co-quaternions.
3. This would suggests that the quaternionic coordinate lines in W can be identified as coordinate curves in O - that space-time surfaces - which are quaternionic/co-quaternionic surfaces for $P^1 = \text{constant}/P^2 = \text{constant}$ lines. One would have a representation of the same thing in two spaces, and if sameness includes also quaternionicity/co-quaternionicity as attributes, then also associativity and co-associativity should hold true.

The most reasonable approach is based on generality. Associativity/quaternionicity means a slicing of octonion space by orthogonal quaternionic and co-quaternionic 4-D surfaces defined by constant value surfaces of octonionic polynomial with real coefficients. This slicing should make sense also for quaternions: one should have a slicing by complex and co-complex (commutative/co-commutative) surfaces and in TGD string world sheets and partonic 2-surfaces assignable to Hamilton-Jacobi structure would define

this kind of slicing. In the case of complex numbers one has a slicing in terms of constant value curves for real and imaginary parts of analytic function and Cauchy-Riemann equations should define the property and co-property. The first guess that the tangent space of the curve is real or imaginary is wrong.

3.2.1 Could associativity and commutativity conditions be seen as a generalization of Cauchy-Riemann conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The "whatever-it-is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: k -linearity with $k = 1, 2, 3$!

One can continue the hierarchy of division algebras by assuming only algebra property by using Cayley-Dickson construction (see <http://tinyurl.com/ybuy1a2k>) by adding repeatedly a non-commuting imaginary unit to the structure already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

3.2.2 Complex curves in real plane cannot have real tangent space

Going from octonions to quaternions to complex numbers, could constant value curves of real and imaginary parts of ordinary analytic function in complex plane make sense? The curves $u = 0$ and $v = 0$ of functions $f(z) = u + iv$, $z = x + iy$ define a slicing of plane by orthogonal curves completely analogous to its octonionic and quaternionic variants. Can one say that the tangent vectors for these curves are real/imaginary? For $u = 0$ these curves have tangent $\partial_x u + i\partial_y u$, which is not real unless one has $f(z) = k(x + iy)$, k real.

Reality condition is clearly too strong. In fact, it is the well-ordering of the points of the 1-dimensional curve, which is the property in question and lost for complex numbers and regained at $u = 0$ and $v = 0$ curves. The reasonable interpretation is in terms of hierarchy of conditions multilinear in the gradients of coordinates proposed above and linear Cauchy-Riemann conditions is the only option in the case of complex plane. What is special in these curves that the tangent vectors define flows which by Cauchy-Riemann conditions are divergenceless and irrotational locally.

Pessimistic would conclude that since the conjecture fails except for linear polynomials in complex case, it fails also in the case of quaternions and octonions. For quaternionic polynomial q^2 the conditions are however satisfied and it turns out that the resulting conditions make sense also in the general case. Optimistic would argue that reality condition is not analogous to commutativity and associativity so that this example tells nothing. Less enthusiastic optimist might admit that the reality condition is a natural generalization to complex case but that the conjecture might be true only for a restricted set of polynomials - in complex case of for $f(z) = kz$, k real. In quaternionic and octonionic case but hopefully for a larger set of polynomials with real coefficients, maybe even all polynomials with real coefficients.

3.2.3 Associativity and commutativity conditions as a generalization of Cauchy-Riemann conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. "Whatever-it-is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: k -linearity with $k = 1, 2, 3$!

One can continue the hierarchy of number fields by assuming only algebra property by adding additional imaginary units as done in Cayley-Hamilton construction (see <http://tinyurl.com/ybuyla2k>) by adding repeatedly a non-commuting imaginary unit to the algebra already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions? Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

One would have also a nice physical interpretation: in the case of quaternions one would have "quaternionic conformal invariance" as conformal invariances inside string world sheets and partonic 2-surfaces in a nice agreement with basic vision about TGD. At the level of octonions would have "quaternionic conformal invariance" inside space-time surfaces and their duals. What selects the preferred commutative or co-commutative surfaces is of course an interesting problem. Is a gauge choice in question? Are these surfaces selected by some special property such as singular character? Or does one have wave function in the set of these surfaces for a given space-time surface?

3.2.4 Could quaternionic polynomials define complex and co-complex surfaces in H_c ?

What about complex and co-complex (commutative/co-commutative) surfaces in the space of quaternions? One would have a slicing of the quaternionic space by pairs of complex and co-complex surfaces and would have natural identification as quaternion/Hamilton-Jacobi structure and relate to the decomposition of space-time to string world sheets and partonic 2-surfaces. Now the condition of associativity would be replaced with commutativity.

1. In the quaternionic case the tangent vectors of the 2-D complex sub-variety would be commuting. Can this be the case for the zero loci real polynomials $P(q)$ with $IM(P) = 0$ or $RE(P) = 0$? In this case the commutativity condition is that the tangent vectors have imaginary parts (as quaternions) proportional to each other but can have different real parts. The vanishing of cross product is the condition now and involves only two vectors whereas associativity condition involves 3 vectors and is more difficult.
2. The tangent vectors of a commutative 2-surface commute: $[t^1, t^2] = 0$. The commutator reduces to the vanishing of the cross product for the imaginary parts:

$$Im(t^1) \times Im(t^2) = 0 \quad . \tag{3.4}$$

3. Expressing z_1^i as a function of z_2^k and using (z_1^i, z_2^k) as coordinates in quaternionic space, the tangent vectors in quaternionic spaces can be written in terms of partial derivatives $\partial z_1^i / \partial z_2^k$ as

$$t_k^i = \left(\frac{\partial z_1^i}{\partial z_2^k}, \delta_k^i \right), \quad (3.5)$$

Here the first part corresponds to $RE(t^i)$ as quaternion and second part to $IM(t^i)$ as quaternion. The condition that the vectors are parallel implies

$$\frac{\partial z_1^i}{\partial z_2^k} = 0. \quad (3.6)$$

At the commutative 2-surface X^2 z_1^1 is constant and z_1^2 is a function of z_2^1 and z_2^2 . One would have a graph of a function $z_1^2 = f_2(z_2^k)$ at X^2 but not elsewhere. One could regard z_1^1 as an extremum of function $z_1^1 = f_1(z_2^k)$. There is no obvious reason why the extremum condition could not make sense. and result might be consistent with the property of being real quaternion polynomial.

How to interpret this result?

1. In the generic case this condition eliminates 1 dimension so that that 2-D surface would reduce to a 1-D curve. This was a cold shower.
2. If one poses constraints on the coefficients of $P(q)$ analogous to the conditions forcing the potential function for say cusp catastrophe to have degenerate extrema at the boundaries of the catastrophe one can get 2-D solution. For these values of parameters the conditions would be equivalent with $RE(P) = 0$ or $IM(P) = 0$ conditions.

The vanishing of the gradient of z_1^1 would indeed correspond in the case of cups catastrophe to the condition for the co-incidence of two roots of the behavior variable x as extremum of potential function $V(x, a, b)$ fixing the control variable a as function of b .

This would pose constraints on the coefficients of P not all polynomials would be allowed. This kind of conditions would realize the idea of quantum criticality of TGD at the level of quaternion polynomials. This option is attractive if realizable also at the level of octonion polynomials. This turns out to be the case.

3. One would thus have two kinds of commutative/co-commutative surfaces. The generic 1-D surfaces and 2-D ones which are commutative/commutative and critical and assignable to string world sheets and partonic 2-surfaces. 1-D surfaces would correspond to fermion lines at the orbits of partonic 2-surfaces appearing in the twistor amplitudes in the interaction regions defined by CDS. 2-D surfaces would correspond to the orbits of fermionic strings connecting point-like fermions at their ends and serving as correlates for bound state entanglement for external fermions arriving into CD. This picture would allow also to understand why just some string world sheets and partonic 2-surfaces are selected.

The simplest manner to kill the proposal is to look for $P = q^2$ and $RE(P(q^2)) = 0$ surface. In this case this condition is indeed satisfied. One has

$$\begin{aligned}
 RE(P) &= X^1 + X^2 I_1 \ , \\
 X^1 &= (z_1^1)^2 - (z_1^2)^2 + (z_2^1)^2 - (z_2^2)^2 \ , \quad X^2 = 2z_1^1 z_1^2 I_1 \ , \\
 IM(P) &= Y^1 + Y^2 I_1 \ , \\
 Y^1 &= (z_2^1 + \overline{z_2^1}) z_1^1 \ , \quad Y^2 = (z_2^2 + \overline{z_2^2}) z_1^2
 \end{aligned} \tag{3.7}$$

$X^2 = 0$ gives $z_1^1 z_1^2 = 0$ so that one has either $z_1^1 = 0$ or $z_1^2 = 0$. $X^1 = 0$ gives for $z_1^1 = 0$ $z_1^2 = \pm \sqrt{(z_2^1)^2 + (z_2^2)^2}$.

The partial derivative $\partial z_1^1 / \partial z_2^k$ is from implicit function theorem - following from the vanishing of the differential $d(RE(P))$ along the surface - proportional $\partial X^1 / \partial z_2^k$, but vanishes as required.

Clearly, the quaternionic variant of the proposal survives in the simplest case its simplest test. 2-D character of the surface would be due to the criticality of q^2 making it possible to satisfy the conditions without the reduction of dimension.

3.2.5 Explicit form of associativity/quaternionicity conditions

Consider now the explicit conditions for associativity.

1. One should calculate the octonionic tangent (normal) vectors t^i for $X = 0$ in associative ($Y = 0$ in co-associative case) and show that there associators $Ass(t^i, t^j, t^k)$ vanish for all possible or all possible combinations i, j, k . In other words, one that one has

$$Ass(t^i, t^j, t^k) = 0 \ , \quad i, j, k \in \{1, \dots, 4\} \ , \quad Ass(a, b, c) \equiv (ab)c - a(bc) \ . \tag{3.8}$$

One can cast the condition to simpler form by expressing t^i as octonionic vectors $t_k^i E^k$:

$$\begin{aligned}
 Ass(E^a, E^b, E^c) &\equiv f^{abcd} E_d \ , \quad a, b, c, d \in \{1, \dots, 7\} \ , \\
 f^{abcd} &= \epsilon^{abe} \epsilon_e^{cd} - \epsilon^{aed} \epsilon_e^{bc} = 2\epsilon^{abe} \epsilon_e^{cd} \ .
 \end{aligned} \tag{3.9}$$

The permutation symbols for a given triplet i, j, k are structures constants for quaternionic inner product and completely antisymmetric (see <http://tinyurl.com/p42tqsq>).. $\epsilon_{ijk} = 1$ is true for the seven triplets 123, 145, 176, 246, 257, 347, 365 defining quaternionic sub-spaces with 1-D intersections. The anti-associativity condition $(E_i E_j) E_k = -(E_i E_j) E_k$ holds true so that one has obtains the simpler expression for f^{ijk} having values ± 2 .

Using this representation $Ass(t^i, t^j, t^k)$ reduces to 7 conditions for each triplet:

$$t_r^i t_s^j t_t^k f^{rstu} = 0 \ , \quad i, j, k \in \{1, \dots, 4\} \ , \quad r, s, t, u \in \{1, \dots, 7\} \ . \tag{3.10}$$

2. If the vanishing condition $X = 0$ or $Y = 0$ is crucial for associativity then every polynomial is its own case to be studied separately and a general principle behind associativity should be identified: the proposal is as a non-linear generalization of Cauchy-Riemann conditions. As the following little calculation shows, the vanishing condition indeed appears as one calculates partial derivatives $\partial z_1^{k)}/\partial z_2^{l)}$ in the expression for the tangent vectors of the surface deduced from the vanishing gradient of X or Y .
3. I have proposed the octonionic polynomial ansatz already earlier but failed to prove that it gives associative tangent or normal spaces. Besides the intuitive geometric argument I failed to notice that the complex 8-D tangent vectors in coordinates $z_1^{k)}$ or $z_2^{k)}$ for complexified space-time surface and coordinates $(z_1^{k)}, z_2^{k)})$ for o have components

$$\frac{\partial o^i}{\partial z_1^k} \leftrightarrow (\delta_k^i, \frac{\partial z_2^{i)}}{\partial z_1^{k)})$$

or

$$(\frac{\partial o^i}{\partial z_2^k}) \leftrightarrow (\frac{\partial z_1^{i)}}{\partial z_2^{k)}, \delta_k^i) .$$

(3.11)

These vectors correspond to complexified octonions O_i given by

$$\delta_k^i E^k + \frac{\partial z_2^{i)}}{\partial z_1^{k)}} E^k E_4 , \tag{3.12}$$

where the unit octonions are given by $(E_0, E_1, E_2, E_3) = (1, I_1, I_2, I_3)$ and $(E_5, E_6, E_7, E_8) = (1, I_1, I_2, I_3)E_4$. The vanishing of the associators stating that one has

4. One can calculate the partial derivatives $\frac{\partial z_2^{k)}}{\partial z_1^{j)}$ explicitly without solving the equations or the complex valued quaternionic components of $RE(P) \equiv X = 0$ or $IM(P) \equiv Y = 0$ (note that X and Y have for complex components labelled by X^i and Y^i respectively).

$$Y^i(z_1^{k)}, z_2^{l)}) = c \in R , \quad i = 1, \dots, 4 , \quad \text{associativity} ,$$

or

$$X^i(z_1^{k)}, z_2^{l)}) = c \in R , \quad i = 1, \dots, 4 , \quad \text{co-associativity} .$$

(3.13)

explicitly and check whether associativity holds true. The derivatives can be deduced from the constancy of Y or X .

5. For instance, if one has $z_2^{k)}$ as function of $z_1^{k)}$, one obtains in the associative case

$$\begin{aligned}
 X^i k + Y_r^i \frac{\partial z_2^i}{\partial z_1^k} &= 0 \\
 X_k^i &\equiv \frac{\partial Y^i}{\partial z_1^k} \quad , \quad Y_k^i \equiv \frac{\partial Y^i}{\partial z_2^k} \quad .
 \end{aligned}
 \tag{3.14}$$

In co-associative case one must consider normal vectors expressible in terms of Y^i so that X is replaced with Y in these equations.

This allows to solve the partial derivatives needed in associator conditions

$$\frac{\partial z_2^i}{\partial z_1^k} = [Y^{-1}]^{ir} X_{rk} \quad .
 \tag{3.15}$$

6. The vanishing conditions for the associators are however multilinear and one can multiply each factor by the matrix Y without affecting the condition so that Y^{-1} disappears and one obtains the conditions for vectors

$$\begin{aligned}
 T_r^i T_s^j T_t^k f^{rstu} &= 0 \quad , \quad i, j, k \in \{1, \dots, 4\} \quad , \quad r, s, t, u \in \{1, \dots, 7\} \quad , \\
 T_k^i &= Y_k^i - X_k^i \quad .
 \end{aligned}
 \tag{3.16}$$

This form of conditions is computationally much more convenient.

How to solve these equations?

1. The antisymmetry of f^{rstu} with respect to the first two indices r, s leads one to ask whether one could have

$$T_r^i T_s^j T_t^k = 0
 \tag{3.17}$$

for the 7 quaternionic triplets. This is guaranteed if one has either $RE(Y)^i_k = \partial Y^i / \partial z_1^k = 0$ (coquaternionic part of T^i) or $IM(Y)^i_k = \partial Y^i / \partial z_2^k = 0$ (co-quaternionic part of T^i) for one member in each triplet.

The study of the structure constants listed above shows that indices 1,2,3 are contained in all 7 triplets. Same holds for the indices appearing in any quaternionic triplet. Hence it is enough to require that three gradients $RE(Y)^i_k = 0$ or $IM(Y)^i_k = 0$ $k \in \{1, 2, 3\}$ vanish. This condition is obviously too strong since already single vanishing condition reduces the dimension of space-time variety to 3 in the generic case and it becomes trivially associative.

Octonionic automorphism group G_2 gives additional basis with their own quaternion triplets and the general condition would be that 3 partial derivatives vanish for a triplet obtained from the basic triplet $\{1, 2, 3\}$ by G_2 transformation. It is not quite clear to me whether the G_2 transformation can depend on position on space-time surface.

2. As noticed, the vanishing of all triplets is an un-necessarily strong condition. Already the vanishing of single gradient $RE(Y)_k^i$ or $IM(Y)_k^i$ reduces the dimension of the surface from 4 to 3 in the generic case. If one accepts that the dimension of associative surface is lower than 4 then single criticality condition is enough to obtain 3-D surface.

In the generic case associativity holds true only at the 4-D tangent spaces of 3-surfaces at the ends of CD (at light-like partonic orbits it holds true trivially in 4-D) and that the twistor lift of Kähler action determines the space-time surfaces in their interior.

In this case one can map only the boundaries of space-time surface by $M^8 - H$ duality to H . At first this looks like a catastrophe but this is not the case. The criticality at these 3-surfaces dictates the boundary conditions and provides a solution to infinite number of conditions stating the vanishing of SSA Noether charges at space-like boundaries. These space-time regions would correspond to the regions of space-time surfaces inside CDs identifiable as interaction regions, where Kähler action and volume term couple and dynamics depends on coupling constants.

The mappability of M^8 dynamics to H dynamics in all space-time regions does not look feasible: the dynamics of octonionic polynomials involves no coupling constants whereas twistor lift of Kähler action involves couplings parameters. The dynamics would be non-associative in the geometric sense in the interior of CDs. Notice that also conformal field theories involve slight breaking of associativity and that octonions break associativity only slightly ($a(bc) = -(ab)c$ for octonionic imaginary units). I have discussed the breaking of associativity from TGD viewpoint in [26].

3. Twistor lift of Kähler action allows also space-time regions, which are minimal surfaces [32] and for which the coupling between Kähler action and volume term vanishes. Preferred extremal property reduces to the existence of Hamilton-Jacobi structure as image of the quaternionic structure at the level of M^8 . The dynamics is universal as also critical dynamics and independent of coupling constants so that $M^8 - H$ duality makes sense for it. External particles arriving into CD via its boundaries would correspond to critical 4-surfaces: I have discussed their interpretation from the perspective of physics and biology in [33].
4. One should be able to produce associativity without the reduction of dimension. One can indeed hope of obtaining 4-D associative surfaces by posing conditions on the coefficients of the polynomial P by requiring that one $RE(Y)_k^i$ or $IM(Y)_k^i$, $i = i_1$ - call it just X_1 - should vanish so that Y^i would be critical as function of z_1^k or z_2^k .

At $X_1 = 0$ would have degenerate zero at the 4-surface. The decomposition of X_1 to a product of monomial factors with root in extension of rationals would have one or more factors appearing at least twice. The associative 4-surfaces would be ramified. Also the physically interesting p-adic primes are conjectured to be ramified in the sense that their decomposition to primes of extension of rationals contains powers of primes of extension. The ramification of the monomial factors is nothing but ramification for polynomial primes in field of rationals in terms of polynomial primes in its extension.

This could lead to vanishing of say one triplet while keeping $D = 4$. This need not however give rise to associativity in which case also second $RE(Y)_i^i$ or $IM(Y)_k^i$, $i = i_2$, call it X_2 , should vanish. The maximal number of X_i would be $n_{max} = 3$. The natural condition consistent with quantum criticality of TGD Universe would be that the variety is associative but maximally quantum critical and has therefore dimension $D = 3$ or $D = 4$. Stronger condition allows only $D = 4$.

These $n \leq 3$ additional conditions make the space-time surface analogous to a catastrophe with $n \leq 3$ behavior variables in Thom's classification of 7 elementary catastrophes with less than 11 control variables [1]. Thom's theory does not apply now since it has only one potential function $V(x)$ (now $n \leq 3$ corresponding to the critical coordinates Y^i !) as a function of behaviour variables and control variables). Also the number of non-vanishing coefficients in the polynomial having

values in an extension of rationals and acting as control variables is unlimited. In quaternionic case the number of potential functions is indeed 1 but the number of control variables unlimited.

5. One should be able to understand the $D = 3$ associative objects - say light-like 3-surfaces or 3-surfaces at the boundaries of CD - as 3-surfaces along which 4-D associative (co-associative) and non-associative (non-co-associative) surfaces are glued together.

Consider a product P of polynomials allowing 3-D surfaces as necessarily associative zero loci to which a small interaction polynomial vanishing at the boundaries of CD (proportional to o^n , $n > 1$) is added. Could P allow 4-D surface as a zero locus of real or imaginary part and containing the light-like 3-surfaces thanks to the presence of additional parameters coming from the interaction polynomial. Can one say that this small interaction polynomial would generate 4-D space-time in some sense? 4-D associative space-time regions would naturally emerge from the increasing algebraic complexity both via the increase of the degree of the polynomial and the increase of the dimension of the extension of rationals making it easier to satisfy the criticality conditions!

There are two regions to be considered: the interior and exterior of CD. Could associativity/co-associativity be possible outside CD but not inside CD so that one would indeed have free external particles entering to the non-associative interaction region. Why associativity conditions would be more difficult to satisfy inside CD? Certainly the space-likeness of M^4 points with respect to the preferred origin of M^8 in this region should be crucial since Minkowski norm appears in the expressions of $RE(P)$ and $IM(P)$.

Do the calculations for the associative case generalize to the co-associative case?

1. Suppose that one has possibly associative surface having $RE(P) = 0$ (as suggested by the emergence of CDs for this choice). One would have $IM(P) = 0$ for dual space-time surface defining locally normal space of $RE(P) = 0$ surface. This would transform the co-associativity conditions to associativity conditions and the preceding arguments should go through essentially as such.

Associative and co-associative surfaces would meet at singularity $RE(P) = IM(P) = 0$, which need not be point in Minkowskian signature (see $P = o^2$ example in the Appendix) and can be even 4-D! This raises the possibility that the associative and co-associative surfaces defined by $RE(P) = 0$ and $IM(P) = 0$ meet along 3-D light-like orbits partonic surfaces or 3-D ends of space-time surfaces at the ends of CD.

2. If $D = 3$ for associative surfaces are allowed besides $D = 4$ as boundaries of 4-surfaces, one can ask why not allow $D = 5$ for co-associative surfaces. It seems that they do not have a reasonable interpretation as a surface at which co-associative and non-co-associative 4-D space-time regions would meet. Or could they in some sense be geometric "co-boundaries" of 4-surfaces like branes in M-theory serve as co-boundaries of strings? Could this mean that 4-D space-time-surface is boundary of 5-D co-associative surface defining a TGD variant of brane with strings world sheets replaced with 4-D space-time surfaces?

What came as a surprise that $P = o^2$ allows 5-D and 6-D surfaces as zero loci of $RE(P)$ or $IM(P)$ as shown in Appendix. The vanishing of the entire o^2 gives 4-D interior or exterior of CD forced also by associativity/co-associativity and thus maximal quantum criticality. This is very probably due to the special properties of o^2 as polynomial: in the generic case the zero loci should be 4-D.

This discussion can be repeated for complex/co-complex surfaces inside space-time surfaces associated with fermionic dynamics.

1. Associativity condition does not force string world sheets and partonic 2-surfaces but they could naturally correspond to commutative or co-commutative varieties inside associative/co-associative varieties.

In the generic case commutativity/co-commutativity allows only 1-D curves - naturally light-like fermionic world lines at the boundaries of partonic orbits and representing interacting point-like fermions inside CDs and used in the construction of twistor amplitudes [28, 29]. There is coupling between Kähler part and volume parts of modified Dirac action inside CDs so that coupling constants are visible in the spinor dynamics and in dynamics of string world sheet.

2. At criticality one obtains 2-D commutative/co-commutative surfaces necessarily associated with external particles quantum critical in 4-D sense and allowing quaternionic structure. String world sheets would serve as correlates for bound state entanglement between fermions at their ends. Criticality condition would select string world sheets and partonic 2-surfaces from the slicing of space-time surface provided by quaternionic structure (having Hamilton-Jacobi structure as H -counterpart).

If associativity holds true and fixed M_c^2 is contained in the tangent space of space-time surface, one can map the M^4 projection of the space-time surface to a surface in $M^4 \times CP_2$ so that the quaternionic tangent space at given point is mapped to CP_2 point. One obtains 4-D surface in $H = M^4 \times CP_2$.

1. The condition that fixed M_c^2 belongs to the tangent space of X_c^4 is true in the sense that the coordinates $z_2^{(k)}$ do not depend on $z_1^{(1)}$ and $z_1^{(2)}$ defining the coordinates of M_c^2 . It is not clear whether this condition can be satisfied in the general case: octonionic polynomials are expected to imply this dependence un-avoidably.

The more general condition allows M_c^2 to depend on position but assumes that M_c^2 :s associated with different points integrate to a family 2-D surfaces defining a family of complexified string world sheets. In the similar manner the orthogonal complements E_c^2 of M_c^2 would integrate to a family of partonic 2-surfaces. At each point these two tangent spaces and their real projections would define a decomposition analogous to that define by light-like momentum vector and polarization vector orthogonal to it. This decomposition would define decomposition of quaternionic sub-spaces to complexified complex subspace and its co-complex normal space. The decomposition would correspond to Hamilton-Jacobi structure proposed to be central aspect of extremals [25].

2. What is nice that this decomposition of M_c^4 (M^4) would (and of course should!) follow automatically from the octonionic decomposition. This decomposition is lower-dimensional analog to that of the complexified octonionic space induced by level sets of real octonionic polynomials but at lower level and extremely natural due to the inclusion hierarchy of classical number fields. Also M_c^2 could have similar decomposition orthogonal complex curves by the value sets of polynomials. The hierarchy of grids means the realization of the coordinate grid consisting of quaternionic, complex, and real curves for complexified coordinates o^k and their quaternionic and complex variants and is accompanied by corresponding real grids obtained by projecting to M^4 and mapping to CP_2 .

Thus these decompositions would be obtained from the octonionic polynomial decomposing it to real quaternionic and imaginary quaternionic parts first to get a grid of space-time surfaces as constant value sets of either real or imaginary part, doing the same for the non-constant quaternionic part of the octonionic polynomial to get similar grid of complexified 2-surfaces, and repeating this for the complexified complex octonionic part.

Unfortunately, I do not have computer power to check the associativity directly by symbolic calculation. I hope that the reader could perform the numerical calculations in non-trivial cases to this!

3.2.6 General view about solutions to $RE(P) = 0$ and $IM(P) = 0$ conditions

The first challenge is to understand at general level the nature of $RE(P) = 0$ and $IM(P) = 0$ conditions. Appendix shows explicitly for $P(o) = o^2$ that Minkowski signature gives rise to unexpected phenomena. In the following these phenomena are shown to be completely general but not quite what one obtains for $P(o) = o^2$ having double root at origin.

1. Consider first the octonionic polynomials $P(o)$ satisfying $P(0) = 0$ restricted to the light-like boundary δM_+^8 assignable to 8-D CD, where the octonionic norm of o vanishes.

(a) $P(o)$ reduces along each light-ray of δM_+^8 to the same real valued polynomial $P(t)$ of a real variable t apart from a multiplicative unit $E = (1 + in)/2$ satisfying $E^2 = E$. Here n is purely octonion-imaginary unit vector defining the direction of the light-ray.

$IM(P) = 0$ corresponds to quaternionicity. If the E^4 ($M^8 = M^4 \times E^4$) projection is vanishing, there is no additional condition. 4-D light-cones M_{\pm}^4 are obtained as solutions of $IM(P) = 0$. Note that M_{\pm}^4 can correspond to any quaternionic subspace.

If the light-like ray has a non-vanishing projection to E^4 , one must have $P(t) = 0$. The solutions form a collection of 6-spheres labelled by the roots t_n of $P(t) = 0$. 6-spheres are not associative.

(b) $RE(PE) = 0$ corresponding to co-quaternionicity leads to $P(t) = 0$ always and gives a collection of 6-spheres.

2. Suppose now that $P(t)$ is shifted to $P_1(t) = P(t) - c$, c a real number. Also now M_{\pm}^4 is obtained as solutions to $IM(P) = 0$. For $RE(P) = 0$ one obtains two conditions $P(t) = 0$ and $P(t - c) = 0$. The common roots define a subset of 6-spheres which for special values of c is not empty.

The above discussion was limited to δM_+^8 and light-likeness of its points played a central role. What about the interior of 8-D CD?

1. The natural expectation is that in the interior of CD one obtains a 4-D variety X^4 . For $IM(P) = 0$ the outcome would be union of X^4 with M_{\pm}^4 and the set of 6-spheres for $IM(P) = 0$. 4-D variety would intersect M_{\pm}^4 in a discrete set of points and the 6-spheres along 2-D varieties X^2 . The higher the degree of P , the larger the number of 6-spheres and these 2-varieties.

2. For $RE(P) = 0$ X^4 would intersect the union of 6-spheres along 2-D varieties. What comes in mind that these 2-varieties correspond in H to partonic 2-surfaces defining light-like 3-surfaces at which the induced metric is degenerate.

3. One can consider also the situation in the complement of 8-D CD which corresponds to the complement of 4-D CD. One expects that $RE(P) = 0$ condition is replaced with $IM(P) = 0$ condition in the complement and $RE(P) = IM(P) = 0$ holds true at the boundary of 4-D CD.

6-spheres and 4-D empty light-cones are special solutions of the conditions and clearly analogs of branes. Should one make the (reluctant-to-me) conclusion that they might be relevant for TGD at the level of M^8 .

1. Could M_{\pm}^4 (or CDs as 4-D objects) and 6-spheres integrate the space-time varieties inside different 4-D CDs to single connected structure with space-time varieties glued to the 6-spheres along 2-surfaces X^2 perhaps identifiable as pre-images of partonic 2-surfaces and maybe string world sheets? Could the interactions between space-time varieties X_i^4 assignable with different CDs be describable by regarding 6-spheres as bridges between X_i^4 having only a discrete set of common points. Could one say that X_i^2 interact via the 6-sphere somehow. Note however that 6-spheres are not dynamical.

2. One can also have Poincare transforms of 8-D CDs. Could the description of their interactions involve 4-D intersections of corresponding 6-spheres?

3. 6-spheres in $IM(P) = 0$ case do not have image under $M^8 - H$ correspondence. This does not seem to be possible for $RE(P) = 0$ either: it is not possible to map the 2-D normal space to a unique CP_2 point since there is 2-D continuum of quaternionic sub-spaces containing it.

3.3 $M^8 - H$ duality: objections and challenges

In the following I try to recall all objections against the reduction of classical physics to octonionic algebraic geometry and against the notion of $M^8 - H$ duality and also invent some new counter arguments and challenges.

3.3.1 Can one really assume distribution of $M^2(x)$?

Hamilton-Jacobi structure means that $M^2(x)$ depends on position and $M^2(x)$ should define an integrable distribution integrating to a 2-D surface. For cosmic string extremals this surface would be minimal surface so that the term "string world sheet" is appropriate. There are objections.

1. It seems that the coefficients of octonionic polynomials cannot contain information about string world sheet, and the only possible choice seems to be that string world sheets and partonic 2-surfaces parallel to it assigned with integrable distribution of orthogonal complements $E^2(x)$ should be coded by quaternionic moduli. It should be possible to define quaternionic coordinates q_i decomposing to pairs of complex coordinates to each choice of $M^2(x) \times E^2(x)$ decomposition of given M_0^4 . Octonionic coordinates would be given by $o = q_1 + q_2 I_4$ where q_i are associated with the same quaternionic moduli. The choice of Hamilton-Jacobi structure would not be ad hoc procedure anymore but part of the definition of solutions of field equations at the level of M^8 .
2. It would be very nice if the quaternionic structure could be induced from a fixed structure defined for M_c^8 once the choice of curvilinear M^4 coordinates is made. Since Hamilton-Jacobi structure [25] involves a choice of generalized Kähler form for M^4 and since quaternionic structure means that there is full S^2 of Kähler structures determined by quaternionic imaginary units (ordinary Kähler form for sub-space $E^8 \subset M_c^8$) the natural proposal is that Hamilton-Jacobi structures is determined by a particular local choice of the Kähler form for M^4 involving fixing of quaternionic imaginary unit fixing $M^2(x) \subset M_0^4$ identifiable as point of S^2 . This might relate closely also to the fixing of twistor structure, which indeed involves also self-dual Kähler form and a similar choice.
3. One can argue that it is not completely clear whether massless extremals (MEs) [25] allow a general Hamilton-Jacobi structure. It is certainly true that if the light-like direction and orthogonal polarization direction are constant, MEs exist. It is clear that if the form of field equations is preserved and thus reduces to contractions of various tensors with second fundamental form one obtains only contractions of light-like vector with itself or polarization vector and these contractions vanish. For spatially varying directions one could argue that light-like direction codes for a direction of light-like momentum and that problems with local conservation laws expressed by field equations might emerge.

3.3.2 Can one assign to the tangent plane of $X^4 \subset M^8$ a unique CP_2 point when M^2 depends on position

One should show that the choice $s(x) \in CP_2$ for a given distribution of $M^2(x) \subset M^4(x)$ is unique in order to realize the $M^8 - H$ correspondence as a map $M^8 \rightarrow H$. It would be even better if one had an analytic formula for $s(x)$ using tangent space-data for $X^4 \subset H$.

1. If $M^2(x) = M_0^2$ holds true but the tangent space $M^4(x)$ depends on position, the assignment of CP_2 point $s(x)$ to the tangent space of $X^4 \subset M^8$ is trivial. When $M^4(x)$ is not constant, the situation is not so easy.
2. The space $M^2(x) \subset M^4(x)$ satisfies also the constraint $M^2(x) \subset M_0^4$ since quaternionic moduli are fixed. To avoid confusion notice that $M^4(x)$ denotes tangent space of X^4 and is different from M_0^4 fixing the quaternionic moduli.

3. $M^2(x)$ determines the local complex subspace and its completion to quaternionic tangent space $M^4(x)$ determines a point $s(x)$ of CP_2 . The idea is that M_0^2 defines a standard reference and that one should be able to map $M^2(x)$ to M_0^2 by G_2 automorphism mapping also the $s(x)$ to a unique point $s_0(x) \in CP_2$ defining the CP_2 point assignable to the point of $X^4 \subset M^8$.
4. One can assign to the point x quaternionic unit vector $n(x)$ determining $M^2(x)$ as the direction of the preferred imaginary unit. The G_2 transformation must rotate $n(x)$ to n_0 defining M_0^2 and acts on s . G_2 transformation is not unique since $u_1 g u_2$ has the same effect for $u_i \in U(2)$ leaving invariant the point of CP_2 for initial and final situation. Hence the equivalence classes of transformations should correspond to a point of 6-dimensional double coset space $U(2) \backslash G_2 / U(2)$. Intuitively it seems obvious that the $s_0(x)$ is unique but proof is required.

3.3.3 What about the inverse of $M^8 - H$ duality?

$M^8 - H$ duality should have inverse in the critical regions of $X^4 \subset M^8$, where associativity conditions are satisfied. How could one construct the inverse of $M^8 - H$ duality in these regions? One should map space-time points $(m, s) \in M^4 \times CP_2$ to points $(m, e) = (m, f(m, s)) \in M^8$. $M_0^4 \supset M_0^2$ parameterized by CP_2 point can be chosen arbitrarily and one can require that it corresponds to some space-time point $(m_0, s_0) \in H$. CP_2 point $s(x)$ characterizes the quaternionic tangent space containing $M^2(x)$ and can choose M_0^2 to be $M^2(x_0)$ for conveniently chosen x_0 . Coordinates x can be used also for $X^4 \subset M^8$.

One obtains set of points $(m, e) = (m(x), f(m(x), s(x))) \in M^8$ and a distribution of 4-D spaces of labelled by $s(x)$. This requires that the 4-D tangent space spanned by the gradients of $m(x)$ and $f(m(x), s(x))$ and characterized by $s_1 \subset CP_2$ for given $M^2(x)$ by using the above procedure mapping the situation to that for M_0^2 is same as the tangent space determined by $s(x)$: $s(x) = s_1(x)$. Also the associativity conditions should hold true. One should have a formula for s_1 as function of tangent vectors of space-time surface in M^8 . The ansatz based on algebraic geometry in M_c^8 should be equivalent with this ansatz. The problem is that the ansatz leads to algebraic functions which cannot be found explicitly. It might be that in practice the correspondence is easy only in the direction $M^8 \rightarrow H$.

3.3.4 What one can say about twistor lift of M^8 and $M^8 - H$ duality?

One can argue that the twistor spaces CP_1 associated with M^4 and E^4 are in no way visible in the dynamics of octonion polynomials and in $M^8 - H$ duality. Hence one could argue that they are not needed for any reasonable purpose. I cannot decide whether this is indeed the case. There I will consider the existence of twistor lift of the M^8 and also the twistor lift $M^8 - H$ duality in the space-time regions, where the tangent spaces satisfy the conditions for the existence of the duality as a map $(m, e) \in M^8 \rightarrow (m, s) \in M^4 \times CP_2$ must be considered. This involves some non-trivial delicacies.

1. The twistor bundles of M_c^4 and E_c^4 would be simply $M_c^4 \times CP_1$ and $E_c^4 \times CP_1$. $CP_1 = S^2$ parameterizes direction vectors in 3-D Euclidian space having interpretation as unit quaternions so that this interpretation might make sense. The definition of twistor structure means a selection of a preferred quaternion unit and its representation as Kähler form so that these twistor bundles would have thus Kähler structure. Twistor lift replaces complex quaternionic surfaces with their twistor spaces with induced twistor structure.
2. In M^8 the radii of the spheres CP_1 associated with M^4 and E^4 would be most naturally identical whereas in $M^4 \times CP_2$ they can be different since CP_2 is moduli space. Is the value of the CP_2 radius visible at all in the classical dynamics in the critical associative/co-associative space-time regions, where one has minimal surfaces. Criticality would suggest that besides coupling constants also parameters with dimension of length should disappear from the field equations. At least for the known extremals such as massless extremals, CP_2 type extremals, and cosmic strings CP_2 radius plays no role in the equations. CP_2 radius comes however into play only in interaction regions

defined by CDs since $M^8 - H$ duality works only at the 3-D ends of space-time surface and at the partonic orbits. Therefore the different radii for the CP_1 associated with CP_2 and E^4 cause no obvious problems.

Consider now the idea about twistor space as real part of octonionic twistor space regarded as quaternion-complex space. One can regard

1. One can regard $CP_1 = S^2$ as the space of unit quaternions and it is natural to replace it with the 6-sphere S^6 of octonionic imaginary units at the level of complexified octonions. The sphere of complexified (by i) unit octonions is non-compact space since the norm is complex valued and this generalization looks neither attractive nor necessary since the projection to real numbers would eliminate the complex part.

The equations determining the twistor bundle of space-time surface can be indeed formulated as vanishing of the quaternionic imaginary part of S^6 coordinates, and one obtains a reduction to quaternionic sphere S^2 at space-time level.

If S^2 is identified as sub-manifold $S^2 \subset S^6$, it can be chosen in very many manners (this is of course not necessary). The choices are parameterized by $SO(7)/SO(3) \times SO(4)$ having dimension $D = 12$. This choice has no physical content visible at the level of H . Note that the Kähler structure determining Hamilton-Jaboci structure is fixed by the choice of preferred direction ($M^2(x)$). If all these moduli are allowed, it seems that one has something resembling multiverse, the description at the level of M^8 is deeper one and one must ask whether the space-time surfaces with different twistorial, octonionic, and quaternionic moduli can interact.

2. The resulting octonionic analog of twistor space should be mapped by $M^8 - H$ corresponds to twistor space of space-time surface $T(M^4) \times T(CP_2)$. The radii of twistor spheres of $T(M^4)$ and $T(CP_2)$ are different and this should be also understood. It would seem that the radius of $T(M^4)$ at $H = M^4 \times CP_2$ side should correspond to that of $T(M^4)$ at M^8 side and thus to that of S^6 as its geodesic sphere: Planck length is the natural proposal inspired by the physical interpretation of the twistor lift. The radius of $T(CP_2)$ twistor sphere should correspond to that of CP_2 and is about 2^{12} Planck lengths.

Therefore the scale of CP_2 would emerge as a scale of moduli space and does not seem to be present at the level of M^8 as a separate scale. M^8 level would correspond to what might be called Planckian realm analogous to that associated with strings before dynamical compactification which is now replaced with number theoretic compactification. The key question is what determines the ratio of the radii of CP_2 scale to Planck for which favored value is 2^{12} [27]. Could quantum criticality determine this ratio?

4 Appendix: o^2 as a simple test case

Octonionic polynomial o^2 serves as a simple testing case. o^2 is not irreducible so that its properties might not be generic and it might be better to study polynomial of form $P(o) = o + po^2$ instead.

Before continuing, some conventions are needed.

1. The convention is that in $M^8 = M^1 \times E^7 E^7$ corresponds to purely imaginary complexified octonions in both octonionic sense and in the sense that they are proportional to i . M^1 corresponds to octonions real in both senses. This corresponds to the signature $(1, -1, -1, -1, \dots)$ for M^8 metric obtained as restriction of complexified metric. For $M^4 = M^1 \times E^3$ analogous conventions hold true.
2. Conjugation $o = o_0 + o_k I_k \rightarrow \bar{o} \equiv o_0 - I_k o_k$ does not change the sign of i . Quaternions can be decomposed to real and imaginary parts and some notation is needed. The notation $q = Re(q) + Im(q)$ seems to be the least clumsy one: here $Im(q)$ is 3-vector.

The explicit expression in terms of quaternionic decomposition $o = q_1 + q_2 I_4$ reads as

$$P(o) = o^2 = q_1^2 - q_2 \bar{q}_2 + (q_1 q_2 + q_2 \bar{q}_1) I_4 . \quad (4.1)$$

o corresponds to complexified octonion and there are two options concerning the interpretation of M^4 and E^4 . M^4 could correspond to quaternionic or co-quaternionic sub-space. I have assumed the first interpretation hitherto but actually the identification is not obvious. This two cases are different and must be treated both.

With these notations quaternionic inner product reads as

$$\begin{aligned} q_1 q_2 &= Re(q_1 q_2) + Im(q_1 q_2) , \\ Re(q_1 q_2) &= Re(q_1) Re(q_2) - Im(q_1) \cdot Im(q_2) , \\ Im(q_1 q_2) &= Re(q_1) Im(q_2) + Re(q_2) Im(q_1) + Im(q_1) \times Im(q_2) . \end{aligned} \quad (4.2)$$

Here $a \cdot b$ denotes the inner product of 3-vectors and $a \times b$ their cross product.

Note that one has real and imaginary parts of octonions as two quaternions and real and imaginary parts of quaternions. To avoid confusion, I will use RE and IM to denote the decomposition of octonions to quaternions and Re and Im for the decomposition of quaternions to real and imaginary parts.

One can express the $RE(o^2)$ as

$$\begin{aligned} RE(o^2) &\equiv X \equiv q_1^2 - q_2 \bar{q}_2 , \\ Re(X) &= Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) , \\ Im(X) &= Im(q_1^2) = 2Re(q_1) Im(q_1) . \end{aligned} \quad (4.3)$$

For $IM(o^2)$ one has

$$\begin{aligned} IM(o^2) &\equiv Y = q_1 q_2 + q_2 \bar{q}_1 \\ Re(Y) &= 2Re(q_1) Re(q_2) , \\ Im(Y) &= Re(q_1) Im(q_2) - Re(q_2) Im(q_1) + Im(q_1) \times Im(q_2) . \end{aligned} \quad (4.4)$$

The essential point is that only $RE(o^2)$ contains the complexified Euclidian norm $q_2 \bar{q}_2$ which becomes Minkowskian of Euclidian norm depending on whether one identifies M^4 as associative or co-associative surface in o_c^8 .

4.1 Option I: M^4 is quaternionic

Consider first the condition $RE(o^2) = 0$. The condition decomposes to two conditions stating the vanishing of quaternionic real and imaginary parts:

$$\begin{aligned} Re(X) &= Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{E^4}(q_2) = 0 , \\ Im(X) &= Im(q_1^2) = 2Re(q_1) Im(q_1) = 0 . \end{aligned} \quad (4.5)$$

$Im(X) = 0$ is satisfied for $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has two options. This gives 1-D line in time direction of 3-D hyperplane as a solution for M^4 factor.

$Re(X) = 0$ states $N_{M^4}(q_1) = N_{E^4}(q_2)$. q_2 coordinate itself is free. $N_{E^4}(q_2)$ is negative so that q_1 must be space-like with respect to the N_{M^4} so that only the solution $Re(q_1) = 0$ is possible. Therefore one has $Re(q_1) = 0$ and $N_{M^4}(q_1) = N_{E^4}(q_2)$.

One can assign to each E^4 point a section of hyperboloid with $t = 0$ hyper-plane giving a sphere and the surface is 6-dimensional sphere bundle like variety! This is completely unexpected result and presumably is due to the additional accidental symmetries due to the octonionicity. Also the fact that o^2 is not irreducible polynomial is a probably reason since for o the surface is 4-D. The addition of linear term is expected to remove the degeneracy.

Consider next the case $IM(o^2) = 0$. The conditions read now as

$$\begin{aligned} Re(Y) &= 2Re(q_1)Re(q_2) = 0 \quad , \\ Im(Y) &= Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) = 0 \quad . \end{aligned} \tag{4.6}$$

Since cross product is orthogonal to the the factors $Im(Y) = 0$ condition requires that $Im(q_1)$ and $Im(q_2)$ are parallel vectors: $Im(q_1) = \lambda Im(q_2)$ and one has the condition $Re(q_1) = \lambda Re(q_2)$ implying $q_1 = \Lambda q_2$. Therefore to each point of E^4 is associated a line of M^4 . The surface is 5-dimensional.

It is interesting to look what the situation is if both conditions are true so that one would have a singularity. In this case $Re(q_1) = 0$ and $Re(q_1) = \lambda Re(q_2)$ imply $\lambda = 0$ so that $q_1 = 0$ is obtained and the solution reduces to 4-D E^4 , which would be co-associative.

4.2 Option II: M^4 is co-quaternionic

This case is obtained by the inspection of the previous calculation by looking what changes the identification of M^4 as co-quaternionic factor means. Now q_1 is Euclidian and q_2 Minkowskian coordinate and $q_2 \bar{q}_2$ gives Minkowskian rather than Euclidian norm.

Consider first $RE(o^2) = 0$ case.

$$\begin{aligned} Re(X) &= Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{M^4}(q_2) = 0 \quad , \\ Im(X) &= Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \quad . \end{aligned} \tag{4.7}$$

$N_{M^4}(q_1) - N_{M^4}(q_2) = 0$ condition holds true now besides the condition $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has also now two options.

1. For $Re(q_1) = 0$ $N_{M^4}(q_1)$ is non-positive and this must be the case for $N_{M^4}(q_2)$ so that the *exterior* of the light-cone is selected. In this case the points of M^4 with fixed N_{M^4} give rise to a 2-D intersection with $Re(q_1) = 0$ hyper-plane that is sphere so that one has 6-D surface, kind of sphere bundle.
2. For $Im(q_1) = 0$ Minkowski norm is positive and so must be corresponding norm in E^4 so that in E^4 surface has future light-cone as projection. This surface is 4-D. The emergence of future light-cone might provide justification for the emergence of CDs and zero energy ontology.

For $IM(o^2)$ the discussion is same as in quaternionic case since norm does not appear in the equations.

At singularity both $RE(o^2)$ and $IM(o^2) = 0$ vanish. The condition $q_1 = \Lambda q_2$ reduces to $\Lambda = 0$ so that $q_1 = 0$ is only allowed. This leaves only light-cone boundary under consideration.

The appearance of surfaces with dimension higher than 4 raises the question whether something is wrong. One could of course argue that associativity allows also lower than 4-D surfaces as associative surfaces and higher than 4-D surfaces as co-associative surfaces. At H -level one can say that one has 4-D

surfaces. A good guess is that this behavior disappears when the linear term is absent and origin ceases to be a singularity.

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