Article

Murphy's Combinatorial Identity

V. Barrera-Figueroa[1], J. López-Bonilla[*2] & R. López-Vázquez[2]

[1]SEPI-UPIITA, Instituto Politécnico Nacional (IPN), México
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, México

Abstract

We employ the Gauss hypergeometric function to study the Murphy's combinatorial identity.

Keywords: Gauss hypergeometric function, Chu-Vandermonde's identity, Gamma function.

1. Introduction

Here we consider the expression:

$$Q(m,n) \equiv \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{k+m}{m}, \qquad m,n = 0,1,\ldots; \tag{1}$$

Murphy [1, 2] proved that:

$$Q(m,n) = 0, \qquad 0 \le m < n\,, \tag{2}$$

and Quaintance-Gould [3] give the value:

$$Q(n,n) = (-1)^n\,, \qquad n \ge 0. \tag{3}$$

In Sec. 2 we show that in the general case [4]:

$$Q(m,n) = (-1)^n \binom{m}{n}, \tag{4}$$

hence the results (2) and (3) are evident from (4).

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, CDMX, México
E-mail: jlopezb@ipn.mx

2. Murphy's identity

From (1):

$$Q(m,n) = \sum_{k=0}^{\infty} t_k\,, \qquad t_k = (-1)^k \binom{n}{k}\binom{k+m}{m},$$

thus $\frac{t_{k+1}}{t_k} = \frac{(k-n)(k+m+1)}{(k+1)^2}$, then:

$$Q(m,n) = {}_2F_1(-n, m+1; 1; 1) = \frac{\Gamma(n-m)}{n!\,\Gamma(-m)}\,, \tag{5}$$

where was applied the Chu-Vandermonde's identity [5]; the relation (5) implies the values (2) and (3).

If $m \geq n$, the expression (5) acquires the form $[N = m - n \geq 0]$:

$$Q(m,n) = \frac{\Gamma(-N)}{n!\,\Gamma(-m)} \overset{[6]}{=} \text{ equation (4), q.e.d.}$$

We emphasize that (4) is valid $\forall\, m, n \geq 0$. The present study of the Murphy's combinatorial identity exhibits the important participation of Gauss hypergeometric function ${}_2F_1$ and of the Euler's gamma function. The property (2) is useful to prove the orthogonality of Laguerre polynomials $L_n(x)[1,2]$.

Received August 12, 2017; Accepted October 14, 2017

References

1. R. Murphy, *On the inverse method of definite integrals, with physical applications,* Trans. Cambridge Phil. Soc. **5** (1833-35) 113-148.
2. J. Mawhin, A. Ronveaux, *Schrödinger and Dirac equations for the hydrogen atom, and Laguerre polynomials,* Arch. Hist. Exact Sci. **64** (2010) 429-460.
3. J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers,* World Scientific, Singapore (2016).
4. J. Riordan, *Combinatorial identities,* John Wiley & Sons, New York (1968).
5. G. F. Andrews, R. Askey, R. Roy, *Special functions,* Cambridge University Press (2000).
6. J. P. Hannah, *Identities for the gamma and hypergeometric functions: an overview from Euler to the present,* Master of Science Thesis, University of the Witwatersrand, Johannesburg, South Africa (2013).