Article

On the Amdeberhan's Combinatorial Identity

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Abstract

We present an elementary proof of a combinatorial identity obtained by Amdeberhan via the Zeilberger's algorithm.

Keywords: Cancellation identity, combinatorial identities.

1. Introduction

Amdeberhan [1] employed the Zeilberger's algorithm [2, 3] to deduce the combinatorial identity:

$$A \equiv \sum_{j=0}^{k+1} \frac{(-1)^{j+1}}{n-j} \binom{k}{j-1} \binom{j}{r} = \frac{(-1)^k}{n-1} \frac{\binom{n}{r}}{\binom{n-2}{k}}, \qquad 0 \le r \le k, \qquad n \ge k+2\,; \tag{1}$$

here we exhibit an elementary proof of (1) without the application of the mentioned algorithm.

2. Amdeberhan's identity

First we observe that $\binom{k}{j-1} = \frac{j}{k+1}\binom{k+1}{j}$, then:

$$A = \frac{1}{k+1} \sum_{j=0}^{k+1} (-1)^j \, \frac{n-j-n}{n-j} \binom{k+1}{j} \binom{j}{r} =$$

$$\frac{1}{k+1}\left[\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j}\binom{j}{r} - n \sum_{j=0}^{k+1} \frac{(-1)^j}{n-j} \binom{k+1}{j}\binom{j}{r}\right], \tag{2}$$

but in [4, 5] we find the relation:

$$\sum_{j=0}^{p} (-1)^j \binom{p}{j}\binom{j}{r} = 0, \qquad r < p\,, \tag{3}$$

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which can be applied to (2) because $r < k+1$, thus:

$$A = -\frac{n}{k+1}\binom{k+1}{r}\sum_{j=r}^{k+1} (-1)^j \binom{k+1-r}{j-r}\frac{1}{n-j}\ , \tag{4}$$

where was used the cancellation identity $\binom{k+1}{j}\binom{j}{r} = \binom{k+1}{r}\binom{k+1-r}{j-r}$ [5].

Hence from (4):

$$A = -\frac{(-1)^r\, n}{k+1}\binom{k+1}{r}\sum_{q=0}^{m}\binom{m}{q}\frac{(-1)^q}{N-q}\ , \qquad N = n-r, \qquad m = k+1-r, \tag{5}$$

but we have the identity [4, 5]:

$$\sum_{q=0}^{m} (-1)^q \binom{m}{q}\frac{1}{N-q} = \frac{(-1)^m}{N\binom{N-1}{m}}\ , \qquad N \geq m+1, \tag{6}$$

which is applicable in (5) because $N \geq m+1$ means $n \geq k+2$, therefore:

$$A = \frac{(-1)^k\, n}{(n-r)\,(k+1)}\ \frac{\binom{k+1}{r}}{\binom{n-r-1}{k+1-r}} = \text{eq. (1)}, \quad \text{q.e.d.}$$

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