

**Report****A Numerical Solution of Ermakov Equation Corresponding to Diffusion Interpretation of Wave Mechanics**Victor Christianto<sup>\*1</sup> & Florentin Smarandache<sup>2</sup><sup>1</sup>Malang Institute of Agriculture (IPM), Malang, Indonesia<sup>2</sup>Dept. of Math & Sciences, Univ. of New Mexico, Gallup, New Mexico, USA**ABSTRACT**

It has been long known that a year after Schrödinger published his equation, Madelung also published a hydrodynamics version of Schrödinger equation. Quantum diffusion is studied via dissipative Madelung hydrodynamics. Initially the wave packet spreads ballistically, then passes for an instant through normal diffusion and later tends asymptotically to a sub-diffusive law. In this paper we will review two different approaches, including Madelung hydrodynamics and also Bohm potential. Madelung formulation leads to diffusion interpretation, which after a generalization yields to Ermakov equation. Since Ermakov equation cannot be solved analytically, then we try to find out its solution with *Mathematica* package. It is our hope that these methods can be verified and compared with experimental data. But we admit that more researches are needed to fill all the missing details.

**Keywords:** Quantum hydrodynamics, quantum diffusion, quantum-classical correspondence, Madelung equation, Ermakov equation, computer algebra solution.

**1. Introduction**

The Copenhagen interpretation of quantum mechanics is guilty for the quantum mystery and many strange phenomena such as the Schrödinger cat, parallel quantum and classical worlds, wave-particle duality, decoherence, collapsing wave function, etc. The Copenhagen interpretation of QM was challenged not only by Schrödinger but also by a large group of physicists led by Albert Einstein who claimed that the quantum mechanical description of the physical reality cannot be considered complete, as shown in their famous EPR paper Einstein, Podolsky and Rosen. They concluded their derivations by stating that “*While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however that such a theory is possible.*” Einstein did not object to the probabilistic description of sub-atomic phenomena in quantum mechanics. However, he believed that this probabilistic representation

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\*Correspondence: Victor Christianto, Malang Institute of Agriculture (IPM), Malang, Indonesia.

[http://researchgate.net/profile/Victor\\_Christianto](http://researchgate.net/profile/Victor_Christianto). Email: victorchristianto@gmail.com

was a technique used to overcome the practical difficulties of dealing with a more complicated underlying physical reality, much in the same way he suggested earlier to deal with Brownian motion.

Many scientists have tried, however, to put the quantum mechanics back on *ontological* foundations. For instance, Bohm proposed an alternative interpretation of quantum mechanics, which is able to overcome some puzzles of the Copenhagen interpretation. He developed further the de Broglie pilot-wave theory and, for this reason, the Bohmian mechanics is also known as the de Broglie-Bohm theory.[2]

Long before the Bohmian mechanics proposal, a year after Erwin Schrödinger published his celebrated equation, Erwin Madelung showed (in 1927) that it can be written in a hydrodynamic form. Madelung's representation has a seemingly major disadvantage by transforming the linear Schrödinger equation into two nonlinear ones. Nonetheless, despite of its additional complexity, the hydrodynamic analogy provides important insights with regard to the Schrödinger equation. Quantum diffusion is studied via dissipative Madelung hydrodynamics. Initially the wave packet spreads ballistically, then passes for an instant through normal diffusion and later tends asymptotically to a sub-diffusive law.

Quantum diffusion (QD) describes a wave packet spreading in a dissipative environment at zero temperature. Since quantum effects are significant for light particles mainly, QD is very essential for electrons, which on the other hand are very important in physics and chemistry. QD has been experimentally observed, however, for muons as well, which are about 200 times heavier than electrons. Studies on electron transport in solids are strongly motivated by the semiconductor industry, exploring nowadays quantum effects on nanoscale.[4]

Another important transport process affected by quantum effects is the diffusion of hydrogen atoms or molecules in metals and on solid surfaces. The quantum tunneling accelerates the hydrogen diffusion, which is essential for many modern technologies for storage and use of hydrogen as a fuel, chemical reagent, etc.[4]

In this paper we will review two different approaches, including Madelung hydro-dynamics and also Bohm potential. It can be shown that Madelung formulation leads to diffusion interpretation, which after a generalization yields to Ermakov equation. Since Ermakov equation cannot be solved analytically, then we try to find out its solution with *Mathematica* package. It is our hope that these methods can be verified and compared with experimental data. For other papers discussing the use of Ermakov equation in QM, see [8]-[12].

Nonetheless, we admit that more researches are needed to fill all the missing details, for example we do not yet discuss comparison between quantum trajectories and classical trajectories such as in Wilson chamber experiments.

## 2. Bohmian quantum potential [2]

The evolution of the wave function of a quantum mechanical system consisting of N particles is supposed to be described by the Schrödinger equation:

$$i\eta\partial_t\psi = \left(-\frac{\eta^2}{2m}\nabla^2 + U\right)\psi. \tag{1}$$

The complex wave function can be presented generally in the polar form:

$$\psi = \sqrt{\rho} \exp\left(\frac{iS}{\hbar}\right), \tag{2}$$

Where  $\rho = |\psi|^2$  is the N-particles distribution density and  $\frac{S}{\hbar}$  is the wave function phase.

Introducing equation (2) into (1) one gets a set of equations:

$$\partial_t\rho = -\nabla\cdot(\rho\nabla S / m), \tag{3}$$

$$\partial_t S + \frac{(\nabla S)^2}{2m} + U + Q = 0, \tag{4}$$

Where quantum potential, Q, is defined as follows:

$$Q = -\frac{\eta^2}{2m^2} \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}. \tag{5}$$

Equation (5) is called Bohmian quantum potential.[2]

## 3. Madelung quantum potential[2]

If one starts with a different assumption that in equation (3) S is the hydrodynamic-like velocity potential, not the mechanical action as suggested by Bohm, then he can arrive at different relations, such as the two equations proposed by Madelung as follows:

$$\partial_t\rho = -\nabla\cdot(\rho V), \tag{6}$$

$$m\partial_t V + mV\cdot\nabla V = -\nabla(U + Q), \tag{7}$$

where

$$V = \nabla S / m. \tag{8}$$

Equations (6) and (7) are known as the Madelung quantum hydrodynamics.[2]

#### 4. Quantum Diffusion and Ermakov equation. Numerical solution

Quantum diffusion is studied via dissipative Madelung hydrodynamics. Initially the wave packet spreads ballistically, then passes for an instant through normal diffusion and later tends asymptotically to a sub-diffusive law.

Now, we start with Madelung equations (6)(7)(8), then introducing now both expressions for  $\rho$  and  $V$  in Eq. (7) yields the following equation:[4]

$$m\partial_t^2\sigma + b\partial_t\sigma = \frac{\eta^2}{4m\sigma^3}, \tag{9}$$

describing the evolution of the root-mean-square displacement  $\sigma$ . Introducing new dimensionless dispersion and time parameters, Eq. (9) acquires the universal form of a dissipative Ermakov equation:

$$\partial_\tau^2\xi + \partial_\tau\xi = \xi^{-3}, \tag{10}$$

where

$$\xi^2 \equiv \frac{2b\sigma^2}{\eta} \tag{11}$$

$$\tau = bt/m. \tag{12}$$

It is known that such an Ermakov equation cannot be solved analytically. In reference [4], solutions have been obtained for some limiting cases. Now we will try to find numerical solution using *Mathematica* package using NDSolve, as follows:

```
ODE=x''[t]+x'[t]-1/x[t]^3==0;
```

```
sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}]
```

```
Plot[x[t]/.sol,{t,-10,10}]
```

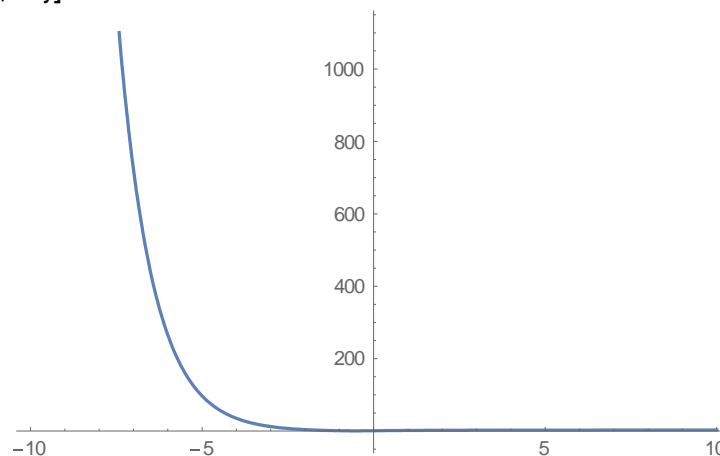


Figure 1. Plot for numerical solution of Ermakov equation

## 5. Discussion & Conclusion

We have discussed two different approaches, including Madelung hydrodynamics and also Bohm potential. Madelung formulation leads to diffusion interpretation, which after a generalization yields to Ermakov equation. Since Ermakov equation cannot be solved analytically, then we try to find out its solution with *Mathematica* package.

We have obtained numerical solution of Ermakov equation corresponding to diffusion interpretation of QM. For other papers discussing the use of Ermakov equation in QM, see [8]-[12]. It is our hope that these methods can be verified and compared with experimental data. Nonetheless, we admit that more researches are needed to fill all the missing details, for example we do not yet discuss comparison between quantum trajectories and classical trajectories such as in Wilson chamber experiments.

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## References

- [1] Robert E. Wyatt. *Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics*. New York: Springer Science + Business Media, Inc., 2005
- [2] Roumen Tsekov. Bohmian mechanics versus Madelung quantum hydrodynamics. *Ann. Univ. Sofia, Fac. Phys. Special Edition* (2012) 112-119 [arXiv 0904.0723]
- [3] \_\_\_\_\_. Complex quantum hydrodynamics with teleportation. *New Adv. Phys.* 8 (2014) 111-121 [arXiv 1301.7537]
- [4] \_\_\_\_\_. Quantum Diffusion. *Phys. Scr.* **83** (2011) 035004 [arXiv 1001.1071]
- [5] V.V. Kulish & J.L. Lage. On the Relationship between Fluid Velocity and de Broglie's Wave Function and the Implications to the Navier–Stokes Equation. *International Journal of Fluid Mechanics Research*, Vol. 29, No. 1, 2002.
- [6] Eugen Merzbacher. *Quantum Mechanics*. 3<sup>rd</sup> ed. New York: John Wiley & Sons, Inc., 1998.
- [7] James M. Feagin. *Quantum Methods with Mathematica*. New York: Springer-Verlag, 1994.
- [8] Erwin Suazo, Sergei K. Suslov, Jose M. Vega-Guzman. The Riccati system and a diffusion type equation. *Mathematics* 2014, 2, 96-118. URL: <http://www.mdpi.com/journal/mathematics>
- [9] P.G. Leach & K. Andriopoulos. The Ermakov equation: a Commentary. *Appl. Anal. Discrete Math.* 2 (2008), 146-157.
- [10] Dieter Schuch. Nonlinear quantum mechanics, complex classical mechanics and conservation laws for closed and open systems. *J. of Phys: Conf. Series* 361 (2012) 012020.
- [11] Jose F. Carinena. A new approach to Ermakov systems and applications in quantum physics. The European Physical Journal Special topics – July 2008. Obtained from ResearchGate.
- [12] Panayotis G. Kevrekidis & Yannis Drossinos. Nonlinearity from linearity: The Ermakov- Pinney equation revisited. *Mathematics and Computers in Simulation* 74 (2007) 196-202.