**Article**

# Some Characterizations of A-Net Minimal Surfaces in the Three-Dimensional Heisenberg Group

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## Abstract

In this paper, we obtain the mean curvature of an A-net surface in the three-dimensional Heisenberg group $\mathsf{H}_3$. Moreover, we give some characterizations of this surface according to Levi-Civita connections of three dimensional Heisenberg group $\mathsf{H}_3$. Finally, we give an example and draw the minimal A-net surface with the help of Mathematica.

**Keywords**: Heisenberg group, minimal surface, Gaussian curvature, A-net surface.

## 1. Introduction

The study of mean curvature extends back to the 18th century. Later, Lagrange looked for a necessary condition for minimizing a certain integral and found the minimal surface equation. Meusnier first defined the term of mean curvature. Important mathematicians such as Gauss and Weierstrass devoted much of their studies to these surfaces. Minimal surfaces have interesting applications in physics such as soap films, hydrodynamic wakes, thermodynamics, fractals, etc.

In this paper, we investigated the minimal surface problem in the three-dimensional Heisenberg group, which is equipped with its standard Carnot-Carathéodory metric. Using a particular surface measure, the characterizations of minimal surfaces in terms of a sub-elliptic partial differential equation and proof of an existence result for the Plateau problem in this setting are made in [13]. Equations for the Gaussian Curvature and for the Laplacian of a minimal surface in the Heisenberg Group $\mathsf{H}^3$ is established in [12]. Next we studied the Gauss map of minimal surfaces in the Heisenberg group Graphic endowed with a left-invariant Riemannian metric and found that the Gauss map of a nowhere vertical minimal surface is harmonic onto the hyperbolic plane $\mathsf{H}^2$. On the contrary, any nowhere antiholomorphic harmonic map onto $\mathsf{H}^2$ is the Gauss map of a nowhere vertical minimal surface in [4]. Some half-space theorems are proven in the Heisenberg and Lie groups that are endowed with their standard left-invariant Riemannian metrics in [5].

The Bonnet problem of determining surfaces in Euclidean three-dimensional space that can allow for at least one nontrivial isometry that preserves principal curvatures is studied in

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Soyuçok, 1995. This problem is considered locally and examined the general case. Later, Bonnet ruled surfaces that admit only one nontrivial isometry that preserve principal curvatures and surfaces whose generators and orthogonal trajectories form a special net called an A-net considered in [8].

In this paper, we studied A-net minimal surfaces in the three-dimensional Heisenberg group. We then give some characterizations of this surface according to Levi-Civita connections of $\mathsf{H}_3$.

## 2. Heisenberg Group $\mathsf{H}_3$

The Heisenberg group historically originates in and still has its strongest ties to quantum physics: there it is a group of unitary operators acting on the space of states induced from those observables on a linear phase space, which are given by linear or by constant functions. So any Heisenberg group is a subgroup of a group of observables in certain simple examples of quantum mechanical systems.

The Heisenberg group $\mathsf{H}_3$ is defined as $\mathsf{R}^3$ with the group operation

$$(x,y,z)*(x_1,y_1,z_1)=\left(x+x_1,y+y_1,z+z_1+\frac{1}{2}(xy_1-x_1y)\right).$$

The left-invariant Riemannain metric on $\mathsf{H}_3$ is given by

$$g=ds^2=dx^2+dy^2+(xdy+dz)^2.$$

The left invariant orthonormal frame on $\mathsf{H}_3$, which is belong to Riemannian metric $g$

$$\mathbf{e}_1=\frac{\partial}{\partial x},\mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\mathbf{e}_3=\frac{\partial}{\partial z}.$$

For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g$,

$$\nabla_{\mathbf{e}_i}\mathbf{e}_j=\frac{1}{2}\begin{bmatrix}0 & \mathbf{e}_3 & -\mathbf{e}_2\\ -\mathbf{e}_3 & 0 & \mathbf{e}_1\\ -\mathbf{e}_2 & \mathbf{e}_1 & 0\end{bmatrix},$$

where the $(i,j)$-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis. Also, we have the Heisenberg bracket relations

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,[\mathbf{e}_1,\mathbf{e}_3]=[\mathbf{e}_2,\mathbf{e}_3]=0.$$

## 3. A-net Surfaces with Constant Gaussian in $\mathrm{H}_3$

In this chapter, we characterized A-net surfaces in the three-dimensional Heisenberg group. Next we obtain the constant Gaussian curvature of this surface. An A-net on a surface such that, when this net is parametrized and the conditions

$$E = G, F = 0, h_{12} = constant. \neq 0, h_{21} = constant. \neq 0$$

are satisfied, it is called an A-net where $E$, $F$, $G$ are the coefficients of the first fundamental form of the surface and $h_{11}$, $h_{12}$, $h_{21}$, $h_{22}$ are the coefficients of the second fundamental form.

Let

$$\varphi(x, y) = \mu(x)\mathbf{e}_1 + \xi(y)\mathbf{e}_2 + \rho(x, y)\mathbf{e}_3 \tag{3.1}$$

be a surface in $(\mathrm{H}_3, g)$. If we take derivatives of the surface, which is given with the parametrization (3.1), we have

$$\begin{aligned} \varphi_x(x, y) &= \mu'(x)\mathbf{e}_1 + \rho_x(x, y)\mathbf{e}_3, \\ \varphi_y(x, y) &= \xi'(y)\mathbf{e}_2 + \rho_y(x, y)\mathbf{e}_3. \end{aligned} \tag{3.2}$$

Then, components of the first fundamental form of the surface are

$$\begin{aligned} E &= \rho_x^2 + \mu'^2(x), \\ F &= \rho_x \rho_y, \\ G &= \xi'^2(y) + \rho_y^2. \end{aligned} \tag{3.3}$$

So, if $F = 0,$ from equations (3.3) we have

$$\rho_y = 0 \text{ or } \rho_x = 0. \tag{3.4}$$

From (3.4), if $\rho_y = 0$, then $\rho(x, y) = \alpha(x)$. If $\rho_x = 0,$ then $\rho(x, y) = \beta(y)$.

**Theorem 3.1.** Let $\varphi(x, y)$ be a surface that is parameterized as (3.1). If $\rho_y = 0,$ then the mean curvature of the surface $\varphi(x, y)$

$$H = \frac{\mu'(x)}{(\alpha'(x) + \mu'(x))^{3/2}}(-\mu''(x)\alpha'(x) + \mu'(x)\alpha''(x)). \tag{3.5}$$

*Proof.* From equations (3.2), we have
$$\mathbf{E}_1 = \mu'(x)\mathbf{e}_1 + \alpha'(x)\mathbf{e}_3,$$

$$\mathbf{E}_2 = \xi'(y)\mathbf{e}_2.$$

Then, components of the first fundamental form of the surface $\varphi(x, y)$ are

$$\begin{aligned} E &= \alpha'^2(x) + \mu'^2(x), \\ F &= 0, \\ G &= \xi'^2(y). \end{aligned} \tag{3.6}$$

So, the induced metric is

$$\tilde{g} = (\alpha'(x) + \mu'^2(x))dx^2 + \xi'^2(y)dy^2.$$

The unit normal vector field of the surface is

$$\mathbf{N} = \frac{1}{\sqrt{\mu'^2(x) + \alpha'(x)^2}}\left(-\alpha'(x)\mathbf{e}_1 + \mu'(x)\mathbf{e}_3\right) \tag{3.7}$$

Then, we have

$$\nabla_{\mathbf{E}_1}\mathbf{E}_1 = \mu'(x)(\mu''(x)\mathbf{e}_1 - \alpha'(x)\mathbf{e}_2 + \alpha''(x)\mathbf{e}_3) \tag{3.8}$$

$$\nabla_{\mathbf{E}_1}\mathbf{E}_2 = \frac{\xi'(y)}{2}\{\alpha'(x)\mathbf{e}_1 + \mu'(x)\mathbf{e}_3\}, \tag{3.9}$$

$$\nabla_{\mathbf{E}_2}\mathbf{E}_1 = \frac{\xi'(y)}{2}\{\alpha'(x)\mathbf{e}_1 - \mu'(x)\mathbf{e}_3\}, \tag{3.10}$$

$$\nabla_{\mathbf{E}_2}\mathbf{E}_2 = \xi'(y)\xi''(y)\mathbf{e}_2 \tag{3.11}$$

So, from (3.7) and (3.8)- (3.11) equations, components of the second fundamental form of the surface are

$$h_{11} = \frac{\mu'(x)}{\sqrt{\mu'^2(x) + \alpha'(x)^2}}\left(\mu'(x)\alpha''(x) - \mu''(x)\alpha'(x)\right), \tag{3.12}$$

$$h_{12} = \frac{\xi'(y)}{2\sqrt{\mu'^2(x) + \alpha'(x)^2}}(\mu'^2(x) - \alpha'(x)^2), \tag{3.13}$$

$$h_{21} = -\frac{1}{2}\xi'(y)\sqrt{\mu'^2(x) + \alpha'(x)^2}, \tag{3.14}$$

$$h_{22} = 0. \tag{3.15}$$

Then, the mean curvature of the surface $\varphi(x,y)$

$$H=\frac{\mu'(x)}{(\alpha'(x)+\mu'(x))^{3/2}}(-\mu''(x)\alpha'(x)+\mu'(x)\alpha''(x)). \tag{3.16}$$

**Theorem 3.2.** Let $\varphi(x,y)$ be a surface that is parameterized as (3.1). If $\varphi(x,y)$ is an A-net surface with $\rho_y=0,$ then $\varphi(x,y)$ is minimal

$$\varphi(x,y)=(\alpha(x)\sqrt{\frac{1+C}{1-C}}+c_1)\mathbf{e}_1+\xi(y)\mathbf{e}_2+\alpha(x)\mathbf{e}_3. \tag{3.17}$$

*Proof*. From (3.13), (3.14), we have

$$\begin{aligned}&h_{12}=constant=A\\ &\Rightarrow\frac{\xi'(y)}{2\sqrt{\mu'^2(x)+\alpha'(x)^2}}(\mu'^2(x)-\alpha'(x)^2)=A\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&h_{21}=constant=B\\ &\Rightarrow-\frac{1}{2}\xi'(y)\sqrt{\mu'^2(x)+\alpha'(x)^2}=B.\end{aligned} \tag{3.19}$$

From (3.18), (3.19), we have

$$\mu(x)=\alpha(x)\sqrt{\frac{1+C}{1-C}}+c_1, \tag{3.20}$$

where $C=\dfrac{A}{B}$ and $c_1$ constant of integration. Then, the mean curvature is

$$H=0. \tag{3.21}$$

So, if $\varphi(x,y)$ is a minimal A-net surface, we have

$$\varphi(x,y)=(\alpha(x)\sqrt{\frac{1+C}{1-C}}+c_1)\mathbf{e}_1+\xi(y)\mathbf{e}_2+\alpha(x)\mathbf{e}_3.$$

So, the proof is complete.

**Corollary 3.3**. Let $\varphi(x,y)$ be A-net surface that is parameterized as (3.1) with $\rho_y=0.$ Then, the Gaussian curvature of the $\varphi(x,y)$ is a non-zero constant.

*Proof.* From equations (3.6), (3.12)-(3.15), the Gaussian curvature is

$$K = \frac{T^2 - 1}{T^2 + 1},$$

where $T = \sqrt{\frac{1+C}{1-C}}$. So,

$$T = 1 \Rightarrow C = 0.$$

If $C = 0,$ then we have $h_{12} = 0,$ which is conrtadicts with the surface is a A-net surface.

**Theorem 3.4.** Let $\varphi(x, y)$ be a surface that is parameterized as (3.1). If $\rho_x = 0,$ then the mean curvature of the surface $\varphi(x, y)$

$$H = \frac{\xi'(y)}{(\alpha'(x) + \xi'(y))^{3/2}}(-\xi''(y)\alpha'(x) + \xi'(y)\alpha''(x)).$$

*Proof.* The proof obtains like theorem 3.2.

**Corollary 3.5**. Let $\varphi(x, y)$ be a surface that is parameterized as (3.1). If $\varphi(x, y)$ is an A-net minimal surface with $\rho_x = 0,$ then

$$\varphi(x, y) = \mu(x)\mathbf{e}_1 + (\beta(y)\sqrt{\frac{1+\tilde{C}}{1-\tilde{C}}} + c_2)\mathbf{e}_2 + \beta(y)\mathbf{e}_3.$$

where $\tilde{C} = \frac{\tilde{h}_{12}}{\tilde{h}_{21}}$ and $c_2$ constant of integration.

**Example 3.6.** Let $\varphi(x, y)$ be a surface in $(\mathrm{H}_3, g)$ that is parameterized as

$$\varphi(x, y) = (\sqrt{3}\cos x, \sin y, \cos x).$$

This surface is a A-net minimal surface in $(\mathrm{H}_3, g)$.

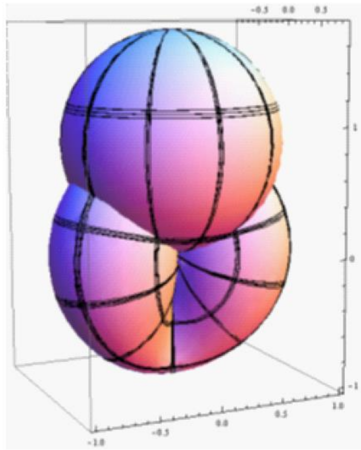

Figure1. A – net minimal surface in $(\mathrm{H}_3, g)$

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