Article

Embedding of R_n into E_{n+1}

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Abstract

We consider an identity, for any R_n embedded into E_{n+1}, between the second fundamental form and the corresponding intrinsic geometry, with special emphasis in the case n = 4, which allows to show that the Pandey-Sharma-Modak spacetime is a counterexample for the Boyer-Plebañski conjecture.

Keywords: Spacetime, class one, local embedding, isometric embedding, conformal, flat space.

1. Introduction

R_n accepts embedding into E_{n+1} if we find the second fundamental form $b_{\mu\nu} = b_{\nu\mu}$ verifying the equations [1, 2]:

$$R_{\mu\nu\alpha\beta} = \varepsilon\,(b_{\mu\alpha}\, b_{\nu\beta} - b_{\mu\beta}\, b_{\nu\alpha}) \qquad \text{Gauss,} \tag{1.a}$$

$$b_{\mu\nu;\alpha} = b_{\mu\alpha;\nu} \qquad \text{Codazzi,} \tag{1.b}$$

where $\varepsilon = \pm 1$, $R_{\mu\nu\alpha\beta}$ is the curvature tensor, and $;\mu$ means covariant derivative. The equation (1.a) allows show the identity [3]:

$$p\, b_{\mu\nu} = \frac{1}{3}\left(2R_{\alpha\mu\beta\nu}\, R^{\alpha\beta} + \frac{1}{2} R_{\mu\alpha\beta\theta}\, R_{\nu}{}^{\alpha\beta\theta} - R\, R_{\mu\nu} - R_{\mu\alpha}\, R_{\nu}{}^{\alpha}\right), \tag{2}$$

where $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = R^{\alpha}{}_{\alpha}$ are the Ricci tensor and the scalar curvature, respectively; besides:

$$p = \frac{2}{3}\,\varepsilon\, b^{\mu\nu} G_{\mu\nu}\,, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu}\,. \tag{3}$$

In Sec. 2 we write (2) for the case n = 4, and the Sec. 3 is dedicated to R_4 conformally flat with an application to the Pandey-Sharma [4]-Modak [5] metric which gives a counterexample to Boyer-Plebañski conjecture [6, 7].

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2. R_4 into E_5

We have the Einstein [8]-Lanczos [9, 10] identities for the double dual of Riemann tensor:

$$ {}^*R^*_{\mu\nu\alpha\beta} = -R_{\mu\nu\alpha\beta} + R_{\mu\alpha}\, g_{\nu\beta} + R_{\nu\beta}\, g_{\mu\alpha} - R_{\mu\beta}\, g_{\nu\alpha} - R_{\nu\alpha}\, g_{\mu\beta} + \frac{R}{2}\left(g_{\mu\beta}\, g_{\nu\alpha} - g_{\mu\alpha}\, g_{\nu\beta}\right), $$
$$ {}^*R^*_{\mu\alpha\beta\theta}\, R_\nu{}^{\alpha\beta\theta} = \tfrac{1}{4} K_2\, g_{\mu\nu}\,, \tag{4} $$

with the invariant [9, 11]:

$$ -24\ \det(b^\mu{}_\nu) = K_2 = {}^*R^*_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} = 4\, I_1 - I_2 - I_3\,, $$
$$ I_1 = R_{\mu\nu}\, R^{\mu\nu}, \qquad I_2 = R^2\,, \qquad I_3 = R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta}\,, \tag{5} $$

where I_3 is the Kretschmann scalar [12, 13], then (2) acquires a simple form for the case $n = 4$ [14-16]:

$$ p\, b_{\mu\nu} = R_{\alpha\mu\beta\nu}\, G^{\alpha\beta} - \tfrac{1}{24} K_2\, g_{\mu\nu}\,, \qquad p^2 = \tfrac{2}{3}\varepsilon\, \left(R_{\alpha\mu\beta\nu}\, G^{\alpha\beta} G^{\mu\nu} + \tfrac{R}{24} K_2\right) \geq 0, \tag{6} $$

thus is clear that p is an intrinsic quantity.

Let's remember that any empty spacetime does not accept embedding into E_5 [2, 14, 17, 18]. We note that in the deduction of (2) and (6) only participates the Gauss equation (1.a), and when $K_2 \neq 0$ we know [19] that (1.a) implies the Codazzi relation (1.b).

3. Conformally flat spacetime

The Weyl tensor is given by [1, 2]:

$$ C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \tfrac{1}{2}\left(R_{\mu\beta}\, g_{\nu\alpha} + R_{\nu\alpha}\, g_{\mu\beta} - R_{\mu\alpha}\, g_{\nu\beta} - R_{\nu\beta}\, g_{\mu\alpha}\right) + \tfrac{R}{6}\left(g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\beta}\, g_{\nu\alpha}\right), \tag{7} $$

thus (6) adopts the structure:

$$ p\, b_{\mu\nu} = C_{\alpha\mu\beta\nu}\, R^{\alpha\beta} + \tfrac{R}{6}\, R_{\mu\nu} - R_{\mu\alpha}\, R^\alpha{}_\nu + \tfrac{1}{6}\left(\tfrac{1}{4} C_2 + \tfrac{5}{2} I_1 - \tfrac{1}{3} I_2\right) g_{\mu\nu}, \ \ C_2 = C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu}, \tag{8} $$

hence for conformally flat 4-spaces of class one:

$$p\, b_{\mu\nu} = \frac{R}{6} R_{\mu\nu} - R_{\mu\alpha} R^{\alpha}{}_{\nu} + \frac{1}{36} (15\, I_1 - 2\, I_2)\, g_{\mu\nu}. \tag{9}$$

Now we can apply (9) to the Pandey-Sharma [4]-Modak [5] geometry:

$$ds^2 = (1 + B(t)\, r^2)^2\, dt^2 - dr^2 - r^2 (d\theta^2 + sin^2\theta\, d\varphi^2), \tag{10}$$

where $B(t)$ is an arbitrary function; if $B = 0$ then (10) gives the Minkowski space. This metric has spherical symmetry and represents a conformally flat perfect fluid distribution with zero density, and with (9) we shall see if is possible its embedding into E_5. From (10) we obtain that $p = 0,\ I_1 = \frac{1}{3} I_2$ and $K_2 = 0$, then:

$$\frac{R}{6}\, R_{\mu\nu} - R_{\mu\alpha}\, R^{\alpha}{}_{\nu} + \frac{1}{2}\, I_2\, g_{\mu\nu} = 0, \tag{11}$$

and the contraction of μ with ν implies that $I_1 = \frac{1}{2}\, I_2$ which contradicts to $I_1 = \frac{1}{3} I_2$ because $R = \frac{12\, B}{1+Br^2} \neq 0$. Therefore, (10) has not class one, thus it is a counterexample for the Boyer-Plebañski conjecture [6, 7]:

"If R_4 is conformally flat with spherical symmetry, then it accepts embedding into E_5". (12)

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