Article

On the Cloitre Formula Involving Harmonic Numbers & the Shifted Factorial

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Abstract

We obtain an identity for the Pochhammer-Barnes symbol which allows one to show the Cloitre's formula.

Keywords: Harmonic numbers, shifted factorial, Pochhammer-Barnes symbol.

1. Introduction

We employ numerical experimentation to find the relation:

$$\sum_{k=n}^{\infty} \frac{1}{(k+1)_m} = \frac{1}{(m-1)\,(n+1)_{m-1}}\ , \quad m = 2, 3, \ldots, \quad n = 1, 2, \ldots \tag{1}$$

with the participation of the Pochhammer [1]-Barnes [2, 3] symbol (shifted factorial [4, 5]):

$$(N)_m \equiv \frac{\Gamma(N+m)}{\Gamma(N)} = \frac{(N+m-1)!}{(N-1)!}. \tag{2}$$

In Sec. 2 we show that (1) implies the identity [6]:

$$\sum_{r=1}^{N} \frac{1}{(r)_m} = \frac{1}{m-1}\left[\frac{1}{(m-1)!} - \frac{N!}{(N+m-1)!}\right], \quad m = 2, 3, \ldots, \quad N = 1, 2, \ldots \tag{3}$$

and in Sec 3 we use (1) to prove the Cloitre formula (2006) [7]:

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)_m} = \frac{1}{(m-1)!\,(m-1)^2}\ , \quad m = 2, 3, \ldots \tag{4}$$

for the important harmonic numbers [8]:

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$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ \dots \ + \frac{1}{k}\,, \qquad k = 1, 2, \dots \tag{5}$$

2. Demonstration of (3)

It is interesting to consider (1) for $n = 1$ and $m = 2$, thus we obtain the known relation of Mengoli [8, 9]:

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)\,(k+2)} = \frac{1}{2}\ ; \tag{6}$$

now we shall apply (1) to show (3), in fact:

$$\sum_{r=1}^{N} \frac{1}{(r)_m} = \frac{1}{(1)_m} + \sum_{k=1}^{N-1} \frac{1}{(k+1)_m} = \frac{1}{m!} + \sum_{k=1}^{\infty} \frac{1}{(k+1)_m} - \sum_{k=N}^{\infty} \frac{1}{(k+1)_m}\,,$$

$$\overset{(1)}{=} \frac{1}{m!} + \frac{1}{(m-1)\,(2)_{m-1}} - \frac{1}{(m-1)\,(N+1)_{m-1}} \overset{(2)}{=} \frac{1}{m!} + \frac{1}{m-1}\left[\frac{1}{m!} - \frac{N!}{(N+m-1)!}\right] = (3), \text{ q.e.d.}$$

We note the inverse option: (3) implies (1) when $N \longrightarrow \infty$.

3. Cloitre formula

The expression (4) is in [7] as a private communication, then here we exhibit that (1) gives an elementary proof of this attractive Cloitre's relation:

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)_m} \overset{(5)}{=} \frac{1}{(2)_m} + \frac{1}{(3)_m}\left(1 + \frac{1}{2}\right) + \frac{1}{(4)_m}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \ \dots,$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k+1)_m} + \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{(k+1)_m} + \frac{1}{3}\sum_{k=3}^{\infty} \frac{1}{(k+1)_m} + \cdots \overset{(1)}{=} \frac{1}{m-1}\left[\frac{1}{(2)_{m-1}} + \frac{1}{2(3)_{m-1}} + \frac{1}{3(4)_{m-1}} + \dots\right] = \frac{1}{m-1}\left[\frac{1}{(2)_{m-1}} + \frac{1}{(2)_m} + \frac{1}{(3)_m} + \ \dots\right] = \frac{1}{m-1}\ \left[\frac{1}{(2)_{m-1}} + \sum_{k=1}^{\infty} \frac{1}{(k+1)_m}\right],$$

$$\overset{(1)}{=} \frac{1}{(m-1)\,(2)_{m-1}}\left(1 + \frac{1}{m-1}\right) \overset{(2)}{=} \frac{1}{(m-1)!\,(m-1)^2}\,, \quad \text{q.e.d.}$$

The deduction of (3) and (4) shows the usefulness of the formula (1).

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References

1. L. Pochhammer, *Über eine klasse von integralen mit geschlossenen integrationskurven*, Math. Ann. **37** (1890) 500-511.
2. E. W. Barnes, *On functions defined by simple hypergeometric series*, Trans. Cambridge Phil. Soc. **20** (1908) 253-279.
3. E. W. Barnes, *A new development of the theory of hypergeometric functions*, Proc. London Math. Soc. **6** (1908) 141-177.
4. E. D. Rainville, *Special functions*, MacMillan Co., New York (1960).
5. W. Koepf, *Hypergeometric summation. An algorithmic approach to summation and special function identities*, Vieweg, Braunschweig / Wiesbaden (1998).
6. http://mathworld.wolfram.com/PochhammerSymbol.html
7. http://mathworld.wolfram.com/HarmonicNumber.html
8. J. Stopple, *A primer of analytic number theory*, Cambridge University Press (2003).
9. P. Mengoli, *Novae quadraturae arithmetica* (1650).