Article

Harmonic Numbers in Terms of Stirling Numbers of the Second Kind

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Abstract

We deduce a formula to generate the harmonic numbers from the Stirling numbers of the second kind.

Keywords: Stirling number, harmonic numbers.

1. Introduction

We know that the Stirling numbers of the first kind allow construct the harmonic numbers [1]:

$$H_n \equiv 1 + \frac{1}{2} + \ \dots + \ \frac{1}{n} = \frac{(-1)^n}{n!} \sum_{k=1}^{n} \ (-1)^k \, k \, S_n^{(k)} \ , \qquad n \geq 1. \tag{1}$$

Here we employ the identity [2]:

$$H_n = \frac{(-1)^{n+1}}{n!} \, S_{n+1}^{(2)} \tag{2}$$

and the Schläfli's expression [1, 3]:

$$S_n^{(n-k)} = (-1)^k \ \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_{k+j}^{[j]} \ , \tag{3}$$

to obtain a formula for H_n in terms of $S_j^{[k]}$, that is, to generate the harmonic numbers via the Stirling numbers of the second kind.

2. Harmonic and Stirling numbers

From (2) and (3):

$$H_n = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{-2}{n+k-1} \binom{2n}{n-k-1} S_{n+k-1}^{[k]} \ , \tag{4}$$

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but we have the property:

$$\binom{-m}{j} = (-1)^j \binom{m+j-1}{j}, \tag{5}$$

in particular $\binom{-2}{j} = (-1)^j\ (j+1)$, therefore:

$$\binom{-2}{n+k-1} = (-1)^{n+k+1}\ (n+k), \tag{6}$$

hence (4) implies the relation:

$$H_n = \frac{1}{n!} \sum_{r=n}^{2n-1} (-1)^{r+1} \binom{2n}{r+1}\ r\, S_{r-1}^{[r-n]}, \quad n \geq 1. \tag{7}$$

We have performed a review of the literature and we have not found the relationship (7). Thanks to Prof. M. Z. Spivey, University of Puget Sound, Tacoma, WA, USA, for his comments about this connection between H_n and Stirling numbers of the second kind.

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