Article

On the Kellner's Formula for Bernoulli Numbers

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Abstract

We show that a relation of Quaintance-Gould implies the Kellner's expression for Bernoulli numbers in terms of Stirling numbers of the second kind.

Keywords: Bernoulli numbers, Kellner's identity.

1. Introduction

In [1] we find the relation:

$$\binom{n+2}{j} B_{n+2-j} = (n+2) \sum_{k=j-1}^{n+1} \frac{1}{k+1} S_{n+1}^{[k]} S_{k+1}^{(j)}, \quad n \geq 0, \quad 1 \leq j \leq n+2, \tag{1}$$

for Bernoulli numbers [2] in terms of Stirling numbers of the first and second kind [3-5].

Here we show that the Kellner's expression [6] for B_m is immediate from (1).

2. Kellner's formula

In (1) we employ $j = 2$ to obtain:

$$B_n = \frac{2}{n+1} \sum_{k=1}^{n+1} \frac{1}{n+1} S_{n+1}^{[k]} S_{k+1}^{(2)}, \tag{2}$$

but we have the following relation for harmonic numbers [7]:

$$H_m = \frac{(-1)^{m+1}}{m!} S_{m+1}^{(2)}, \qquad m \geq 1, \tag{3}$$

hence (2) implies the Kellner's expression [6] to generate Bernoulli numbers from Stirling numbers of the second kind:

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$$B_n = -\frac{2}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k+1} \, k! \; H_k \, S_{n+1}^{[k]} \,, \quad n \geq 0. \tag{4}$$

We note that with the known properties [1, 3, 8]:

$$S_{n+1}^{[0]} = 0, \quad S_{n+1}^{(1)} = (-1)^n \, n! \,, \quad \sum_{r=0}^{n+1} S_{n+1}^{[r]} S_r^{[2]} = \binom{0}{n-1}, \quad \sum_{k=0}^{n} (-1)^k \; k! \; S_n^{[k+1]} = \delta_{n1} \,,$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \, k! \; S_n^{[k]} = B_n \,, \qquad S_{n+1}^{[k]} = S_n^{[k-1]} + k \, S_n^{[k]} \,, \qquad S_{n+1}^{(k)} = S_n^{(k-1)} - n \, S_n^{(k)} \,, \tag{5}$$

it is possible to deduce the Luschny's formula [9] to construct Bernoulli numbers via harmonic numbers:

$$B_n = \sum_{k=0}^{n} H_{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^n \,, \qquad n \geq 0, \tag{6}$$

as an alternative to (4).

Now in (4) we use the Euler's relation [1]:

$$S_m^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \, j^m \binom{k}{j}, \tag{7}$$

and we apply (6) to obtain the representation:

$$(n-1) \, B_n = -2 \sum_{k=0}^{n} \frac{1}{k+2} \, H_{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^n, \qquad n \geq 0. \tag{8}$$

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