Article

Four-dimensional Tensor Identities

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Abstract

We show that an identity of Lovelock, for the conformal tensor in four dimensions, allows to motivate the Edgar's identity which is important in the deduction of the wave equation for the Lanczos spintensor.

Keywords: Weyl tensor, Lanczos potential, tensor identities, four-space.

1. Introduction

The wave equation for the Lanczos generator [1-3] is important in general relativity [4, 5], and in its deduction participates the following Edgar's identity [6]:

$$L^{\mu\rho\lambda}\, W_{\mu\rho\lambda[\alpha}\, g_{\beta]\nu} + 2\, W_{\mu\nu\rho[\alpha}\, L_{\beta]}{}^{\mu\rho} + \frac{1}{2} L^{\mu\rho}{}_{\nu}\, W_{\mu\rho\alpha\beta} = 0, \quad (1)$$

valid only in four dimensions, for arbitrary tensors verifying the properties:

$$L_{\mu\nu\alpha} = -L_{\nu\mu\alpha}\,, \qquad L_{\mu\nu\alpha} + L_{\nu\alpha\mu} + L_{\alpha\mu\nu} = 0, \qquad L_{\mu\nu}{}^{\nu} = 0, \qquad W^{\mu}{}_{\nu\alpha\mu} = 0,$$

$$W_{\mu\nu\alpha\beta} = -W_{\nu\mu\alpha\beta} = -W_{\mu\nu\beta\alpha}\,, \qquad W_{\mu\nu\alpha\beta} + W_{\mu\alpha\beta\nu} + W_{\mu\beta\nu\alpha} = 0, \quad (2)$$

also satisfied by the Lanczos potential [1] and the conformal tensor [7].

Edgar [6] showed his identity employing the generalized Kronecker delta [8]; in Sec. 2 we use a result of Lovelock [9] to motivate the expression (1).

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2. Edgar's identity

Lovelock [9] obtained the following four-dimensional identity:

$$W_{[ab}{}^{[cd}\delta_{p]}^{q]} \equiv 0, \tag{3}$$

that is:

$$\delta_{[a}^{q}\, W_{b]p}{}^{cd} + \delta_{[a}^{c}\, W_{b]}{}^{dq} - \delta_{[a}^{d}\, W_{b]p}{}^{cq} + \delta_{p}^{[c}\, W_{ab}{}^{d]q} + \tfrac{1}{2}\delta_{p}^{q}\, W_{ab}{}^{cd} = 0; \tag{4}$$

now we multiply (4) by $W_{cd}{}^{pr}$ to deduce the relation:

$$W_{\mu\nu\alpha\beta}\, W^{\mu\nu\tau\lambda} + 2\, W_{\mu\nu\gamma}{}^{\lambda}\, W^{\mu\nu\gamma}{}_{[\alpha}\, \delta_{\beta]}^{\tau} + 4\, W^{\lambda}{}_{\mu\nu[\alpha}\, W_{\beta]}{}^{\mu\nu\tau} = 0, \tag{5}$$

where $W_{\mu\nu\alpha\beta}$ has all symmetries of the Weyl tensor.

If in (5) we contract α with λ, we obtain the Lanczos identity [10-12]:

$$W_{abcd}\, W^{abcr} = \tfrac{1}{4}\, W_2\, \delta_d^r\,, \qquad W_2 \equiv W^{\mu\nu\alpha\beta}\, W_{\mu\nu\alpha\beta}\,, \tag{6}$$

hence (5) acquires the form [13]:

$$W_{\mu\nu\alpha\beta}\, W^{\mu\nu\tau\lambda} - 4\, W_{\mu[\alpha}{}^{\tau\nu}\, W_{\beta]}{}^{\mu\lambda} - \tfrac{1}{4}\, W_2\, \delta_{\alpha\beta}^{\tau\lambda} = 0. \tag{7}$$

Finally, the Edgar's identity (1) is immediate from (5) if we multiply it by an arbitrary vector A_λ and we introduce the tensor:

$$L^{\mu\nu\tau} \equiv W^{\mu\nu\tau\lambda}\, A_\lambda\,, \tag{8}$$

which verifies the properties (2).

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References

1. C. Lanczos, *The splitting of the Riemann tensor*, Rev. Mod. Phys. **34**, No. 3 (1962) 379-389.
2. R. Illge, *On potentials for several classes of spinor and tensor fields in curved spacetimes,* Gen. Rel. Grav. **20**, No. 6 (1988) 551-564.
3. G. Ares de Parga, O. Chavoya, J. López-Bonilla, *Lanczos potential,* J. Math. Phys. **30**, No. 6 (1989) 1294-1295.
4. P. Dolan, C. W. Kim, *The wave equation for the Lanczos potential*, Proc. Roy. Soc. London A**447** (1994) 557-575.

5. S. B. Edgar, A. Höglund, *The Lanczos potential for the Weyl curvature tensor: existence, wave equation and algorithms*, Proc. Roy. Soc. London A**453** (1997) 835-851.
6. S. B. Edgar, *The wave equations for the Lanczos tensor/spinor, and a new tensor identity*, Mod. Phys. Lett. A**9** (1994) 479-482.
7. J. Plebañski, A. Krasiñski, *An introduction to general relativity and cosmology*, Cambridge University Press (2006).
8. D. Lovelock, H. Rund, *Tensors, differential forms, and variational principles*, Dover, New York (1989).
9. D. Lovelock, *Dimensionally dependent identities*, Proc. Cambridge Phil. Soc. **68** (1970) 345-350.
10. C. Lanczos, *A remarkable property of the Riemann-Christoffel tensor in four dimensions*, Ann. of Math. **39**, No. 2 (1938) 842-850.
11. D. Lovelock, *The Lanczos identity and its generalizations*, Atti. Accad. Naz. Lincei Rend. **42** (1967) 187-194.
12. M. Novello, J. Duarte de Oliveira, *On dual properties of the Weyl tensor*, Gen. Rel. Grav. **12**, No. 11 (1980) 871-880.
13. S. B. Edgar, *Four-dimensional tensor identities of low order for the Weyl and Ricci tensors*, Gen. Rel. Grav. **31**, No. 3 (1999) 405-411.