Article

Variational Principle for $py'' + qy' + ry = \phi$

J. López-Bonilla[*], G. Posadas-Durán & O. Salas-Torres

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Lindavista 07738, CDMX, México

Abstract

The Lagrangian associated to $p(x)y'' + q(x)y' + r(x)y = 0$ accepts a variational symmetry which allows one to obtain the complete solution of this differential equation if we know one solution for the corresponding homogeneous equation.

Keywords: Variational symmetry, linear, differential equation, second order.

1. Introduction

We study the differential equation:

$$p(x)\, y'' + \, q(x)\, y' + \, r(x)\, y = \, \phi, \tag{1}$$

if we know the solution $y_1(x)$ of its homogeneous equation:

$$p\, y_1'' + \, q\, y_1' + \, r\, y_1 = \, 0; \tag{2}$$

the relation (1) is the Euler-Lagrange expression of the variational principle $\delta \int_{x_1}^{x_2} L\, dx = 0$, such that:

$$L = \left(y'^2 - \frac{r}{p} y^2 + \frac{2\phi}{p}\, y \right) \exp\left(\int^x \frac{q(\xi)}{p(\xi)}\, d\xi \right). \tag{3}$$

In Sec. 2 we show that the infinitesimal transformation:

$$\tilde{y}(x) = y(x) + \varepsilon\, y_1(x), \qquad \varepsilon \ll 1, \tag{4}$$

is a symmetry of the Lagrangian (3), which leads to the general solution of (1).

[*] Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, CDMX, México
E-mail: jlopezb@ipn.mx

2. Variational symmetry

We employ (4) into (3) to deduce:

$$\tilde{L} = L + \frac{d}{dx}\left[2\varepsilon\, y\, y_1' \exp\left(\int^x \frac{q}{p}\, d\xi\right)\right] - \frac{2\varepsilon}{p} y\,(p y_1'' + q y_1' + r y_1) \exp\left(\int^x \frac{q}{p}\, d\xi\right) + 2\varepsilon\, A,$$

with:

$$A \equiv \frac{y_1}{p}\, \phi\, \exp\left(\int^x \frac{q}{p}\, d\xi\right), \tag{5}$$

then from (2):

$$\tilde{L} = L + \frac{d}{dx}\left[2\varepsilon\, y\, y_1' \exp\left(\int^x \frac{q}{p}\, d\xi\right)\right] + 2\varepsilon\, A, \tag{6}$$

which represents a variational symmetry if A is an exact derivative.

In fact, the analysis of A suggests multiply (1) by $\frac{y_1}{p}\exp\left(\int^x \frac{q}{p}\, d\xi\right)$, thus we obtain the interesting relation:

$$\frac{d}{dx}\left[(y_1 y' - y_1'\, y) \exp\left(\int^x \frac{q}{p}\, d\xi\right)\right] = \frac{y_1}{p}\, \phi \exp\left(\int^x \frac{q}{p}\, d\xi\right), \tag{7}$$

therefore the transformation (4) is a symmetry of (3). From (7), two integrations imply the solution of the homogeneous equation:

$$y_2(x) = y_1(x) \int^x \frac{e^{-\int^\eta \frac{q}{p} d\xi}}{[y_1(\eta)]^2}\, d\eta, \tag{8}$$

and the particular solution:

$$y_p = y_2(x) \int^x \frac{y_1 \phi}{p} e^{\int^x \frac{q}{p} d\xi}\, d\eta - y_1(x) \int^x \frac{y_2 \phi}{p} e^{\int^x \frac{q}{p} d\xi}\, d\eta, \tag{9}$$

in harmony with the Lagrange's variation of parameters [1, 2].

We emphasize that (5) gives the factor $\frac{y_1}{p} \exp(\int^x \frac{q}{p}\, d\xi)$ to reduce (1) to an exact derivative, in complete agreement with [3, 4].

Received January 19, 2017; Accepted February 19, 2017

References

1. M. V. Soare, P. P. Teodorescu, I. Toma, *Ordinary differential equations with applications to mechanics,* Springer, Berlin (2007).
2. J. López-Bonilla, D. Romero-Jiménez, A. Zaldívar-Sandoval, *Variation of parameters via the Riccati equation*, Prespacetime Journal **7**, No. 8 (2016) 1217-1219.
3. M. D. Greenberg, *Application of Green's function in science and engineering,* Prentice-Hall, New Jersey (1971).
4. C. Lanczos, *Linear differential operators*, Dover, New York (1997).