Article

BMS Supertranslations as Coset States & SLOCC

Lawrence B. Crowell [1]

Abstract

The algebra of BMS supertranslations and SLOCC for state entanglement of black holes are found to share the same Lax pair structure. In addition this system is a reduced quotient system of variables or states that describes conformal currents. The quotient group and space structure gives Hermitian symmetric space structure to entanglements that under a low energy decomposition recovers $AdS_n \sim CFT_{n-1}$ correspondence and F-theory.

Keywords: Supertranslations, BMS, SLOCC, entanglement, Black Hole, Lax pair.

1 BMS and gravitational memory

A black hole will exhibit a metric back reaction with the emission of Hawking radiation. This metric back reaction due to $M \rightarrow M - \delta M$ results in the emission of gravitons or weak gravity waves. As a result shifts in the holographic qubit content of the stretched horizon release information in a spacetime form, which in turn imparts a memory or information stamp on test masses. The general form of this theory involves Bondi-Metzner-Sachs (BMS) supertranslations[1]. This could indicate one mechanism for how information is conserved in black hole quantum mechanics. What will be illustrated in this paper is how the theory of BMS supertranslations is a form of the SLOCC coset structure for quantum entanglements on black holes[2]. The equivalency between the two is then an argument for how information that reaches $\mathcal{I}^+$ is contain on the event horizon of the black hole.

It is illustrative for physical understanding to consider a linearized form of gravitational memory. In gravitational wave detection this is most likely to be the form gravitational waves and any memory effect will be observed. Gravitational memory from a physical perspective is the change in the spatial metric of a surface according to[3]

$$\Delta h_{+.\times} ~=~ \lim_{t\rightarrow\infty} h_{+,\times}(t) ~-~ \lim_{t\rightarrow-\infty} h_{+,\times}(t).$$

Here $+$ and x refer to the two polarization directions of the gravity wave. See [4] for a more complete treatment. Let us suppose we treat the gravity wave as a linear form of diphoton or colorless state with two gluons, where each photon (or gluon) has a generic state,

$$|\Psi_{+,\times}\rangle ~=~ |h_+(t) ~+~ ih_\times(t)\rangle ~=~ \sum_{l=2}^\infty\sum_{m=-l}^l Y_{lm}[(\theta,\phi)\,(|\uparrow\rangle_+ ~+~ |\downarrow\rangle_+) ~-~ i(\theta,\phi)\,(|\uparrow\rangle_\times ~+~ |\downarrow\rangle_\times)],$$

where the arrows indicate the polarization directions according to their respective axes. The matrix element $H_{+,\times} ~=~ |\Psi_{+,\times}\rangle\langle\Psi_{+,\times}|$ describes the interaction of the gravity wave with a quantum particle. This expanded out is

$$H_{+,\times} ~=~ \sum_{l,l'=2}^\infty\sum_{m=-l}^l\sum_{m'=-l'}^{l'} Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)\,[(|\uparrow\uparrow\rangle_{+\times} ~+~ |\downarrow\downarrow\rangle_{+\times}) ~+~ i\,(|\uparrow\downarrow\rangle_{+\times} ~-~ |\downarrow\uparrow\rangle_{+\times})]$$

[1] Correspondence: Lawrence B. Crowell, PhD, Alpha Institute of Advanced Study, 2980 FM 728 Jefferson, TX 75657 and 11 Rutafa Street, H-1165 Budapest, Hungary. Tel.: 1-903-601-2818 Email: lcrowell@swcp.com.

This tensor operation sets to zero terms like $|\rangle_{++}$ and $|\rangle_{\times\times}$ as unphysical states. This matrix contains a gravity wave term $|\uparrow\uparrow\rangle_{+\times}\ +\ |\downarrow\downarrow\rangle_{+\times}$ plus a scalar term $|\uparrow\downarrow\rangle_{+\times}\ -\ |\downarrow\uparrow\rangle_{+\times}$ that again we set to zero.

One might object to this in that there is an operator for the gravity wave, but we do not have states. However, in the Choy-Jamilkowsky isomorphism [5][6] there is a correspondence between states and operators (really operations) $|\Psi\rangle\langle\Psi'|\ \rightarrow\ |\Psi\rangle\otimes|\Psi'\rangle$ which for the Hilbert space is $H\ \rightarrow\ H\otimes H$. Consequently the matrix pertains directly to states in this scheme. We then have the matrix

$$H_{+,\times}\ =\ \sum_{l,l'=2}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)\left(|\uparrow\uparrow\rangle_{+\times}\ +\ |\downarrow\downarrow\rangle_{+\times}\right),$$

which corresponds to a state vector $|\psi_{+,\times}\rangle$. This is a linear form of gravitational memory and its quantum mechanical analogue.

2 BMS supertranslations and gravitational memory at $\mathcal{I}^+$

We start by looking at the Bondi metric

$$ds^2\ =\ -(1-2m_B/r)du^2-2dudr+2\gamma_{z\bar z}r^2dzd\bar z+rC_{zz}dz^2+rC_{\bar z\bar z}d\bar z^2+D^zC_{zz}dudz+D^{\bar z}C_{\bar z\bar z}dud\bar z+\ \dots. \tag{2.1}$$

The simplest part of the Bondi metric is

$$ds^2\ =\ -du^2\ -\ 2dudr\ +\ 2\gamma_{z\bar z}r^2dzd\bar z,$$

which is just the Minkowski metric for a two sphere with the $z,\ \bar z$ coordinates. The term m_B is the mass term, and the mass is the so called Bondi mass and source of mass-energy propagating out to $\mathcal{I}^+$, where the coordinate u is defined. The terms C_{zz} and $C_{\bar z\bar z}$ determine Weyl curvature terms for gravitational wave propagating out. An Einstein field equation is[7]

$$D_{\bar z}^2C_{zz}\ -\ D_z^2C_{\bar z\bar z}\ =\ 0,$$

which gives the simple solution $C_{zz}\ =\ -2D_z^2C(z,\ \bar z)$. Here $C(z,\ \bar z)$ is a scalar potential. The change in this potential is a change in Weyl curvature with the passage of a gravitational wave.

The curvature has two parts. The source terms define the Bondi mass, and the rate this changes in time, or the parameter u at $\mathcal{I}^+$ is

$$\frac{\partial m_B}{\partial u}\ =\ \frac{1}{4}(D_zD_zN^{zz}\ +\ cc)\ -\ \frac{1}{4}N_{zz}N^{zz}\ +\ 4\pi GT_{uu}(matter)|_{r\ \rightarrow\ \infty},$$

where $N_{zz}\ =\ \partial_uC_{zz}$ is the Bondi news[7]. The term $\frac{1}{4}N_{zz}N^{zz}$ is the flux of gravitational radiation. The stationary spacetime has $M,\ Q,\ J$ as Noether charges or conserved quantities. We have with that system a single set of local symmetries. These are the Lorentz or Poincare symmetries on a spatial surface, which is defined on i_0 since all spatial surfaces contact there. However, we know that this is an idealization of eternal black holes. The emission of gravitational radiation or the time variation of the mass, or Bondi mass, is a signature that something is more general.

The translations in the variables $u,\ r,\ z.\ \bar z$, are[7]

$$\begin{aligned}u\ &\rightarrow\ u\ +\ f(z,\ \bar z)\\ r\ &\rightarrow\ r\ -\ D^zD_zf(z,\ \bar z)\\ z\ &\rightarrow\ z\ +\ \frac{1}{r}D^zf(z,\ \bar z),\end{aligned} \tag{2.2}$$

for a function $f(z,\ \bar z)$, and the translation for $\bar z$ evident. A general infinitesimal supertranslation can be easily seen to then be from these

$$\xi\ =\ f(z,\ \bar z)\frac{\partial}{\partial u}\ +\ D^zD_zf(z,\ \bar z)\frac{\partial}{\partial r}\ -\ \frac{1}{r}\left(D^zf(z,\ \bar z)\frac{\partial}{\partial z}\ +\ c.c.\right).$$

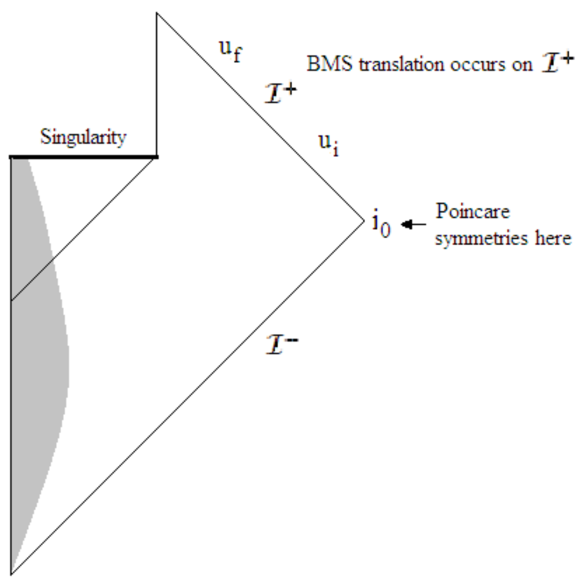
We may then compute the Lie derivative of the Bondi mass and Weyl curvature and the Bondi mass

$$\mathcal{L}_\xi m\ =\ f(z,\ \bar z)\frac{\partial m}{\partial u,},$$

where $\partial m/\partial u$ given above, and the Weyl curvature

$$\mathcal{L}_\xi C_{zz}\ =\ f(z,\ \bar z)N_{zz}\ -\ 2D_z^2f(z,\ \bar z).$$

The supertranslations are additive, where for ξ and ξ' we have additive properties $\xi''\ =\ \xi\ +\ \xi'$, and these are generators of a group with $g\ =\ e^{p^a\xi_a}$ with the index summed over $u,\ r,\ z,\ \bar z$. Clearly for $g'\ =\ e^{p^a\xi'_a}$ we have that $gg'\ =\ g'g$ from the additive nature of the supertranslation vectors and that metric elements are commutative. This is then an infinite dimensional abelian group.



What are these symmetries? They are Noether charges can be computed from the change in the Weyl curvature. We then have for the u_i to u_f on $\mathcal{I}^+$

$$\Delta C\ =\ \int_{\mathbb{S}^2}dzd\bar zG(z,\ \bar z, z',\ \bar z')\int_{u_i}^{u_f}du\left(T_{uu}\ -\ \frac{1}{4}N_{zz}N^{zz}\ +\ c.c.\right)\ =\ f(z,\ \bar z) \qquad (2.3)$$

with the endpoints of the integration u_i and u_f on $\mathcal{I}^+$. This is the integrated gravitational memory at $\mathcal{I}^+$. This integrates a current that flows to $\mathcal{I}^+$ parameterized by u. We may consider this charge as

$$Q\ =\ \int d\Sigma^a(h_{bc}\nabla^a\phi^{bc}\ -\ \phi^{bc}\nabla^ah_{bc}) \qquad (2.4)$$

for the field given by the infinitesimal supertranslations $\phi^{bc} = [\nabla^b, \xi^c]$. The term in the integration is a current, which may equivalently evaluate ΔC.

Now look at the metric near the event horizon, where instead of the time variable u along $\mathcal{I}^+$ we have v along the horizon with

$$ds^2 = -\Phi dv^2 - e^{2\beta} dv dr + g_{AB}(dx^A + X^A dv)(dx^B + X^B dv),$$

where ϕ is a potential and (Φ, β, g, X) are arbitrary functions which means by themselves contain no particular symmetries. The coordinate v is normal to u along the horizon and thus metric terms with dv are zero on the horizon. Near the horizon the BMS translations are

$$\xi = f(z, \bar{z})\frac{\partial}{\partial v} + \left(\int dr - X^A \partial_A f\right)\frac{\partial}{\partial r} + \left(\int dx - g^{AB}\partial_A f\right)\frac{\partial}{\partial X^B},$$

where the limits on the integral are from the horizon r_h to r or $r \to \infty$ at $\mathcal{I}^+$. This is the near field description that shares the same information as the above translation for information sent to $\mathcal{I}^+$.

These calculations of the supertranslation in spacetime are classical. We need of course to think of these quantum mechanically. The theory modulo local translations motives quotient algebras and qubit entanglement-black hole correspondence. The above indicates how the BMS supertranslations in the classical case at $\mathcal{I}^+$ correspond to those on the event horizon. Below a connection via the Lax-pair system, the quotient algebra of qubit-black hole correspondence is found to generalize the bracket structure of the BMS algebra. This means that the BMS supertranslation may be quantized. An elementary model for quantization is presented.

While the metric elements are commutative, we see below in equation 5 that the metric elements and their derivatives obey a set of Poisson brackets, and their generalization as Dirac brackets, that are nonzero. For Poisson brackets $\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$ in the quantization limit. This means that the infinite dimensional albelian algebra is a classical realization of a set of symmetries that are quantum mechanical. For a group formed from quantum mechanical operators as generators is no longer abelian. The conformal nature of this translation means its quantization of has a Virasoro realization, and it then can have an infinite number of symmetries. The discussion on the equivalency between BMS supertranslations and the qubit/black hole correspondence works with the bracket structure of the BMS algebra and works within the context of this nonabelian algebra.

There is a classical-quantum correspondence at work. The infinite dimensional nature of the abelian group of supertranslations and the corresponding quantization, or pre-quantization, that has a Virasoro and Kac-Moody Lie algebraic point to a correspondence of information. The elements of the Virasoro algebra $L_m = \sum_n X^a_{-n} X^a_{m+n}$, are formed from Kac-Moody Lie algebra

$$[X^a_m, X^b_n] = C^{abc} X^c_{m+n} + 2Dm\delta_{m+n}\delta^{ab}.$$

Here C^{abc} the structure constant of some $GL(n, \mathbb{C})$ and D is the dimension of the anomaly cancellation term. The Kac-Moody algebra is an infinite dimensional extension of the Lie algebra. The physical argument is then made that the classical symmetries of the abelian BMS translations have a one to one correspondence with the quantum symmetries. The quantum (or pre-quantum) symmetries point to the structure of quantum spacetime, at least in the context of a many body theory that approximates the exact theory.

3 BMS supertranslations and entanglement equivalency with black holes

Strominger appeals to the Bondi-Metzner-Sachs derivation on how a change in metric, which requires a time-like Killing vector, is most often a fundamental change in the metric[7]. This often involves a change in the Weyl curvature with $C_{\alpha i\beta i}g^{\alpha\beta}$ as the determinant of change in the deviation between two test masses

$$\Delta x^i \ = \ C^{\alpha i\beta i}U_\alpha U_\beta x_i$$

with the passage of a gravity wave, or the metric back reaction from Hawking radiation. From Strominger's paper is the image

We can see that gravitational waves pass through a region that contains two particles separated by a length ℓ. These gravitational waves region asymptotic infinity $\mathcal{I}^+$ at u_i and u_f and this determines the change in the separation of the two particles. This is a imprint of the information in the gravity waves. The derivative of the reduced Weyl curvature $C_{ii} \ = \ C^{\alpha i\beta i}g_{\alpha\beta}$ along the time direction $\partial_t C_{ii} \ = \ N_{ii}$ is the so called Bondi news. For the index i so that $x^i \ = \ z$ along the wave direction, the change with time is then an impulse the gravity wave delivers to the test masses. This is the "report" or information imparted to the moving particles that a gravitational wave passed across their geodesic.

n Stephen Hawking realized information in the Bondi News encodes the metric back reaction on a black hole. A black hole emits Hawking radiation and what is less discussed is that it has a metric back reaction. If the black hole mass is adjusted according to $m \ \to \ m \ + \ \delta m$ the metric changes accordingly, and this is transmitted to the outside world in the form of a gravitational wave. Hawking proposes the information that leaves an event horizon is equivalent to the information a gravitational wave delivers to the outside world to $\mathcal{I}^+$ which occurs with the irreversible change in the geodesic motion of the test masses.

The Bondi-Metzner-Sachs result indicates that the vacuum before and after the gravtational radiation passes are inequivalent. We may think of this in a rubber sheet analogue, where the gravitational radiation has stretched or compressed the rubber sheet in the asymptotic limit in such as manner that the two classical spatial surfaces are not identical. This is the case with a black hole, where the Boulware vacuum near the horizon is irreversibly changed with the emission of boson in Hawking radiation. The conjecture by Hawking is that the two changes are identified, which then means that information apparently lost from the black hole is recovered at $\mathcal{I}^+$ or in part by Δx^i in geodesic motion of masses. Since test masses serve the role of a gravitational wave detector, and they are finite in number or distribution it is clear that the total information is not recoverable. Some is deposited on the test masses, such as by changing their configurations or even quantum states (qubits), but other information makes it way to $\mathcal{I}^+$.

It is then worth considering that the entanglement structure of a black hole is then identified with the entanglement structure of the spacetime at $\mathcal{I}^+$ plus the radiation emitted. Of course this radiation and metric back reaction gravitational wave occur at a huge distance by the the time a black hole evaporates. It takes 10^{76} years for a stellar mass black hole to evaporate, and up to 10^{120} years for the potentially largest black holes (such as future ones formed from coalescence of galaxy clusters $\simeq \ 10^{40}$ years from now) to evaporate. This means this information is around a distance equivalent to the time of decay. Consequently there are cosmological considerations, such as cosmological and particle event horizons, to take into account in a full theory. As an idealization we consider this information at $\mathcal{I}^+$.

Now let us illustrate some calculations that demonstrate how to get qubit entanglement structure of black holes as equivalent to gravitational memory at $\mathcal{I}^+$. This structure can be deduced from the dynamics of the test masses. The bracket structure for the BMS algebra of supertranslations[8]

$$\begin{aligned}
\{C_{\bar z\bar z}(u,\ z,\ \bar z),\ C_{z'z'}(u',\ z',\ \bar z')\} \ &= \ 8\pi G\theta(u \ - \ u')\delta^2(z \ - \ z')\gamma_{z\bar z}\\
\{C(z,\ \bar z),\ C_{z'z'}(u',\ z',\ \bar z')\} \ &= \ 8D^2_{z'}Sln(|z \ - \ z'|^2)\\
\{N_{\bar z\bar z}(z,\ \bar z),\ C_{z'z'}(u',\ z',\ \bar z')\} \ &= \ 16GD^2_{z'}Sln(|z \ - \ z'|^2)\\
\{N_{\bar z\bar z}(z,\ \bar z),\ C(z',\ \bar z')\} \ &= \ 16GSln(|z \ - \ z'|^2).
\end{aligned} \tag{3.1}$$

is an example of how symplectic and Noetherian symmetries interplay with each other. Here

$$S \;=\; \frac{(z \;-\; z')(\bar{z} \;-\; \bar{z}')}{(1 \;+\; z\bar{z})(1 \;+\; z'\bar{z}')}$$

and the Bondi news is the boundary $N_{zz} \;=\; \partial_t C_{zz}$. The additional commutators occur from the definition

$$D_{z',\bar{z}'}\{N_{z,\bar{z}},\; C(z',\bar{z}')\} \;=\; \lim_{u\to\infty}\{N_{u,z,\bar{z}},\; C(u,\; z',\bar{z}')\}$$

and similarly the Bondi news is computated by integrating the same Poisson bracket from $-\infty$ to ∞. The phase space for this system is then

$$\Gamma := \{C_{zz}(u,\; z,\; \bar{z}), C_{\bar{z}\bar{z}}(u,\; z,\; \bar{z}), C(z,\; \bar{z}), N(z,\; \bar{z})\}, \tag{3.2}$$

where the first two are coordinate-like and the last two are boundaries or on the tangent space.

Now focus attention largely on the metric coefficient $rC_{zz}dz^2$, or equivalently $rC_{\bar{z}\bar{z}}d\bar{z}^2$. We consider this as a metric for $\mathbb{S}^2$ embedded in the spacetime. We let $(dx_i/x_i)(dx_j/x_j) \;=\; g_{ik}dy^idy^k$, and in particularly

$$\left(\frac{dx_2}{x_2}\right)^2 \;=\; C_{zz}dz^2,\; \left(\frac{dx_3}{x_3}\right)^2 \;=\; C_{\bar{z}\bar{z}}d\bar{z}^2.$$

With elements $x_2 \;=\; exp(\int\sqrt{C_{zz}}dz)$, $x_3 \;=\; exp(\int\sqrt{C_{\bar{z}\bar{z}}}d\bar{z})$. Using techniques in [9]this is extended to the other elements as well in an interval

$$ds^2 \;=\; g_{ij}dy^idy^j \;\rightarrow\; h_{ij}(xdx)_i(xdx_j)$$

with the flat metric

$$[\mathbf{h}] \;=\; \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix} \tag{3.3}$$

where similarly we have coefficients for the other two coordinates. For the moment we ignore these and consider the x_2 and x_3 metric components and treat the rest of the metric as flat. We then have

$$ds^2 \;=\; x^{-1}\cdot dxx^{-1}\cdot dx,$$

with the group $SL(2,\;\mathbb{C})$ acting transitively on the set X, so that any $x \;\in\; X$ is transformed into any other point $x \;\rightarrow\; gxg^{-1} =\; gxg^\dagger$. Thus if the initial point is a unit then $x \;=\; gg^\dagger$. This representation is not unique, for an element k of the subgroup $K \;\subset\; G$, which is $SU(2)$ for $SL(2,\;\mathbb{C})$ multiplying on the right leaves x unchanged. We then see that the space $X \;=\; G/K =\; SL(2,\;\mathbb{C})/SU(2)$.

With this reduced Bondi metric we have the geodesic equation

$$\frac{D}{ds}\left(x^{-1}\cdot\frac{dx}{ds}\right) \;=\; 0 \;or\; \frac{D}{ds}\left(\frac{dx}{ds}\cdot x^{-1}\right) \;=\; 0,$$

such that $x^{-1}\cdot dx/ds$ are tangent vectors to G as well as $dx/ds\cdot x^{-1}$, and their sum however is tangent to the space X so then

$$\frac{D}{ds}\left[\frac{1}{2}\left(x^{-1}\cdot\frac{dx}{ds} \;+\; \frac{dx}{ds}\cdot x^{-1}\right)\right] \;=\; 0,$$

for inside the bracket is $d(x^{-1}x)/ds \;=\; d(gg^\dagger)/ds$ as motion of a unit. The term in the square brackets is equal to $x^{-1}A + Ax^{-1}$ which implies that $A \;=\; 0$[9]. The invariant quantity in then $L \;=\; \sqrt{C_{zz}} \;+\; \sqrt{C_{\bar{z}\bar{z}}}$, which is implicitly a two component object or vector with components $\sqrt{C_{zz}}$ and $\sqrt{C_{\bar{z}\bar{z}}}$. This geodesic equation then defines all paths on X which for $g\,\in SL(2,\mathbb{C})$ is

$$x(s)\;=\;ge^{2\alpha s}g^\dagger.$$

When reduced to the diagonal form is

$$x(s)\;=\;u(s)e^{2aQ(s)}u^\dagger(s),$$

for $u(s)\;\in\;SU(2)$ as the angular variable, similar to a Bloch sphere, and with $Q(s)$ as a diagonal matrix of entries that projects the matrix to the reduced sphere. We now have

$$\frac{1}{2}\left(x^{-1}\cdot\frac{dx}{ds}\;+\;\frac{dx}{ds}\cdot x^{-1}\right)\;=\;2auLu^\dagger$$

for

$$L\;=\;\frac{DQ}{ds}\;+\;\frac{1}{4a}\left[e^{-2aQ}Me^{2aQ}\;-\;e^{2aQ}Me^{-2aQ}\right] \tag{3.4}$$

with $M\;=\;u^{-1}du/ds$, that is a form of angular velocity. The geodesic equation

$$\frac{D}{ds}\left[\frac{1}{2}\left(x^{-1}\cdot\frac{dx}{ds}\;+\;\frac{dx}{ds}\cdot x^{-1}\right)\right]\;=\;D(auLu^\dagger)\;=\;2au\left(\frac{DL}{ds}u^\dagger\;+\;\frac{Du}{ds}Lu^\dagger\;+\;uL\frac{Du^\dagger}{ds}\right)$$

$$=2au\left(\frac{DL}{ds}\;+\;i\big[M,\;L\big]\right)u^\dagger. \tag{3.5}$$

The geodesic condition is then equal to $DL/ds\;+\;i[M,\;L]\;=\;0$, which is a covariant form of the Lax pair equation. This is equivalent to the dynamical equation given by the infinitesimal supertranslation vector [10]

$$\xi\;=\;f\frac{\partial}{\partial u}\;+\;(D^zD_zf\;+\;C.C.)\frac{\partial}{\partial r}\;-\;\frac{1}{r}\left(D^zf\frac{\partial}{\partial z}\;+\;C.C\right)$$

We desire to examine more than just this truncated metric. This means there are metric elements that are the tangent, where for $x\;\in\;X$ there exists $y\;\in\;T^*X$, the cotangent bundle. The symplectic form

$$\Omega\;=\;Tr(dp\wedge d(x^{-1}))\;=\;-Tr(x^{-1}dp\wedge x^{-1}dx),$$

is invariant under the action of $G:x\;\rightarrow\;gxg^{-1}$, $p\;\rightarrow\;gpg^{-1}$. It is not hard to see that the Hamiltonian $H(x,p)\;=\;\frac{1}{2}Tr(px^{-1}px^{-1})$ with the equations of motion

$$\frac{dp}{ds}\;=\;\frac{\partial H}{\partial x^{-1}}\;=\;px^{-1}p,\frac{dx^{-1}}{ds}\;=\;\frac{\partial H}{\partial p}\;\rightarrow\;\frac{dx}{ds}\;=\;p,$$

where in general the differentials are made into covariant derivatives. A similar analysis as above results in the Lax pairs

$$L\;=\;\frac{DQ}{ds}\;+\;\frac{1}{4a}\left(e^{-2aQ}Me^{2aQ}\;+\;e^{2aQ}Me^{-2aQ}\right)$$

$$M\;=\;y^{-1}\frac{dy}{ds}, \tag{3.6}$$

with $x(s)\;=\;y(s)e^{2aQ(s)}y(s)^\dagger$. Then

$$\frac{D}{ds}\left(x^{-1}\frac{dx}{ds}\right)\;=\;2ay\left(\frac{DL}{ds}\;+\;i[M,\;L]y^\dagger\right), \tag{3.7}$$

with the Lax pair equation $DL/ds + i[M,\ L] = 0$. The equation gives us that $Dx/ds = p = (du/ds)(dx/du)$, and at $r \rightarrow \infty$

$$\frac{du}{ds} \simeq 1 - \sqrt{C_{zz}}\frac{dz}{ds} + C.C. = 1 - x_2\frac{dx_2}{ds} - x_3\frac{dx_3}{ds}.$$

The Hamiltonian is the invariant $H = I^2 \quad Tr(L^2)$ for $L = \sqrt{C_{zz}} + \sqrt{C_{\bar{z}\bar{z}}}$. The dynamical evolution in u is then according to

$$L' = L\frac{du}{ds} \simeq L = C_{zz} + C_{\bar{z}\bar{z}},$$

and so the Hamiltonian is $H = C_{zz}^2 + C_{\bar{z}\bar{z}}^2$. The dynamical equation above gives the Bondi news $N_{zz} = \partial_u C_{zz}$ and $N_{\bar{z}\bar{z}} = \partial_u C_{\bar{z}\bar{z}}$.

The coset Stochastic Local Classical Communications (SLOCC) metric construction in Duff et al is a Lax pair system. This is seen in [2], with the $D = 3$ or 4 case the Hamiltonian is $p^2 = \gamma_{3ij}\phi'^i\phi'^j$, where the prime means time derivative. The coset representation of G/H, for H a Cartan MCS group contains elements $L \in G/H$ defines vielbeins and connection one forms $L^{-1}dL \in \mathfrak{g}$, for $\mathfrak{g}$ the algebra or derivation of G. This is

$$L^{-1}dL = d\phi^i V_i^A T_A$$

for V_i^A the vielbein and $T_A \in \mathfrak{g}$. We may in greater generality appeal to the more symmetric matrix $M = L\eta L^T$. For the $g \in G$ we have that

$$M \rightarrow gMg^{-1} = gL\eta L^T g^{-1}$$

$$= (gLg^{-1})g\eta g^{-1}(gL^T g^{-1})$$

which means $L' = gLg^{-1}$. It is then easy to see that $L' - L = gLg^{-1} - L$. Then for $g = e^{i\sigma t}$ and $t \rightarrow \delta t$ we can easily see that

$$L - L' = i[\sigma,\ L]\delta t,$$

which for $L' - L = \delta L$ and $L' = dL/dt$,

$$L' = i[\sigma,\ L]. \tag{3.8}$$

This illustrates that the black hole qubit correspondence is consistent with a Lax pair system.

The entanglement equivalency with black holes is with SLOCC and multi-partite entanglements according to quotient groups. These groups are on Hermitian symmetric spaces. This is a space with an isometry σ_x defined at a point $x \in H$ such that $\sigma^2 = id$. This means that x is an isolated fixed point. This group of isometries acts transitively on X, as well as on the identity component of the group G such that $H = G/K$. For $\mathfrak{h}$ and $\mathfrak{k}$ the algebras that generate H and K, the algebra of G is

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$$

where it is easy to see these algebras are

$$[\mathfrak{h},\ \mathfrak{h}] \subset \mathfrak{h},\ [\mathfrak{h},\ \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k},\ \mathfrak{k}] \subset \mathfrak{h}. \tag{3.9}$$

that form the Cartan decomposition of $\mathfrak{g}$. For the algebra $\mathfrak{g}$ the subgroup $\mathfrak{k}$ share the nilpotent orbits $\mathcal{N}$ such that

$$\frac{\mathcal{N}\cap\mathfrak{g}}{G} \sim \frac{\mathcal{N}\cap\mathfrak{k}}{H},$$

so the adjonit orbits of G are diffeomorphic to nilpotent of orbits in H. This is the Kostant Sekiguchi theorem.

We may then choose $\mathfrak{g} \ = \ so(8) \ = \ so(4) \ + \ so(4) + (\mathbf{4}, \ \mathbf{4})$, where $\mathfrak{h} \ = \ so(4) \ + \ so(4)$. This may be decomposed further into $so(8) \ = \ sl(2) \ + \ sl(2) \ + \ sl(2) \ + \ sl(2) + (\mathbf{2}, \ \mathbf{2}, \ \mathbf{2}, \ \mathbf{2})$. This leads to

$$\frac{\mathcal{N} \cap so(8)}{SO(8)} \ \sim \ \frac{\mathcal{N} \cap sl(2) \ + \ sl(2) \ + \ sl(2) \ + \ sl(2)}{SL(2, C)^4}.$$

For $\mathcal{N} \ = \ 8$ supersymmetry we may have

$$\mathfrak{g} \ = \ e(8) \ = \ so(16) \ + \ \mathbf{128} \ = \ \mathfrak{h} \ + \ \mathfrak{k}$$

with the same Cartan decomposition.

The STU model with $E_{7(7)}$ and maximal compact subgroup $SU(8)$ has mass and $4 \ + \ 4$ electric and magnetic charges with 28 gauge potentials transforms as the $\mathbf{56}$ of $E_{7(7)}$. The quartic invariant I_4 of $E_{7(7)}$ defines the Bekenstein-Hawking entropy $S \ = \ \pi\sqrt{|I_4|}$. The Cartan decomposition has

$$e_{7(7)} \ = \ su(8) \ + \ \mathbf{70}$$

for the gauge fields plus 70 scalar fields. The $SU(8)$ give the 28 electric and 28 magnetic gauge potentials as the $\mathbf{56}$ of $E_{7(7)}$ with the gauge condition. The $\mathbf{56}$ is represented by further decompositions as $SL(2, \mathbb{R})^7$ and these are a permutation on the states $(\mathbf{2}, \ \mathbf{2}, \ \mathbf{1}, \mathbf{2}, \ \mathbf{1}, \ \mathbf{1}, \ \mathbf{1})$. This decomposition for $e_{7(7)}$ will operate as well for the BMS translation.

The root system of any Lie algebra with $\alpha \ \in \ E^n$, for E^n a Euclidean space, there are reflections of the form

$$r(\beta) \ = \ \beta \ - \ \frac{(\beta. \ \alpha)}{(\alpha, \ \alpha)}\alpha$$

with the matrix of reflections $c_{ij} \ = \ (\alpha_i, \ \alpha_j)/(\alpha_j, \ \alpha_j)$. Root vectors defined in this space E_α in the Chevelley basis obey

$$[E_\alpha, \ E_{-\alpha}] \ = \ H_\alpha, \ [H_\alpha, \ E_\beta] \ = \ \beta(H_\alpha)E_\beta$$
$$[E_\alpha, \ E_\beta] \ = \ N_{\alpha+\beta}E_{\alpha+\beta} \tag{3.10}$$

The Lax pair operators may then be written as

$$L \ = \ \sum_{j=1}^{n}(b_jH_j \ + \ a_j(e_j \ + \ e_{-j}))\,, \ M \ = \ \sum_{j=1}^{n} a_j\,(e_j \ - \ e_{-j}) \tag{3.11}$$

for

$$H_j \ = \ \frac{2\alpha_j}{(\alpha_j, \ \alpha_j)}$$

The invariants of the system are $I^k \ = \ \frac{1}{2}L^k$ and the Hamiltonian $I^2 \ = H \ = \ \frac{1}{2}(L, \ L)$ is then

$$H \ = \ \frac{1}{2}\sum_{j,k}(\alpha_j, \ \alpha_k)b_jb_k \ + \ \sum_j g_j^2 a_j^2. \tag{3.12}$$

in this equation we have $a_j \ = \ f(\alpha_j, q_j)$ and $b_j \ = \ (\beta, p)$ and $g_j \ = \ (E_j, \ E_{-j})$. For the function f such that $f(\alpha_j, q_j) \ = \ exp(\alpha_j, q_j)$ this is the Toda lattice. It is easy to see that Poisson brackets between a_j and b_j give the dynamical equations

$$\dot{b}_j \ = \ \{H, \ b_j\} \ = \ -2(a_{j-1}^2 \ - \ a_j^2), \ \dot{a}_j \ = \ \{H, \ a_j\} \ = \ a_j(b_j \ - \ b_{j+1}).$$

The Toda lattice system is a canonical transformation of coordinate and position variables.

The Hamiltonian for E_7 and SU(8) are given by

$$A_3:\ H\ =\ \frac{1}{2}\sum_{j=1}^{3}p_j\ +\ e^{q_1\ -\ q_2}\ +\ e^{q_2\ -\ q_3}$$

$$E_7:\ H\ =\ \frac{1}{2}\sum_{j=1}^{8}p_j\ +\ e^{q_1\ -\ q_2}\ +\ e^{q_2\ -\ q_3}\ \dots\ e^{q_5\ -\ q_6}\ +\ e^{-q_1\ -\ q_2}\ +\ e^{-q_1\ +\ q_2\ \cdots\ q_7\ -\ q_8} \tag{3.13}$$

$$=\ H_{A_3}\ +\ \frac{1}{2}\sum_{j=4}^{8}p_j\ +\ e^{q_3\ -\ q_4}\ +\ e^{q_4\ -\ q_5}\ +\ e^{q_5\ -\ q_6}\ +\ e^{-q_1\ -\ q_2}\ +\ e^{-q_1\ +\ q_2\ \cdots\ q_7\ -\ q_8},$$

where we can see the $SL(4,\mathbb{C})\ =\ SU(8)$ Hamiltonian embeded in the E_7 Hamiltonian. More justification for these Hamiltonians is given in the final section. The splitting of the Hamiltonians is then an instance of the rank 7 coset $G_4/H_4\ = E_{7(7)}/SU(8)$ associated with the rank 8 $G_3/H_3\ = E_{8(8)}/SO(16)$. This system may be expressed according to Lax pairs. This is the $\mathcal{N}\ =\ 8\ \frac{1}{8}$ supersymmetric BPS attractor point, and the $\mathcal{N}\ =\ 2\ \frac{1}{2}$ supersymmetric BPS attractor point due to the nilpotency condition on the algebra $\mathfrak{k}$, which is a Lax pair system.

The BMS transformations has the principal commutator

$$\{C_{\bar{z}\bar{z}}(t,z,d\bar{z}),C_{z'z'}(t',z',d\bar{z}')\}\ =\ 8\pi G\theta(t\ -\ t')\delta^2(z\ -\ z')\gamma_{z\bar{z}}.$$

We may then write $\hat{C}$ and $\hat{C}^\dagger$ as the quantized operator form of these variables, where now these are forms of the Verlinde operators [11]. These operators act both exterior and interior to a black hole.

4 Toda lattice realizations

The Toda lattice is a classical system similar to the Ising model. It is a chain of particles that have momenta and linked by potentials that have Lie algebraic content. One motivation for this system is celestial mechanics, the other is solid state physics. For Newtonian mechanics was have the basic Hamiltonian $H\ =\ \frac{1}{2}p^2\ -\ kr^{-1}$. The substitutions $u\ =\ 1/r$ and $u\ =\ e^x$ the Hamiltonian is

$$H\ =\ \left(\frac{1}{2}p^2\ -\ e^{4x}\right)e^{-2x}.$$

If we let the term in parentheses left of e^{-2x} be the reduced Hamiltonian H' then

$$\dot{p}\ =\ -\frac{\partial H}{\partial x},\ \dot{p}'\ =\ -\frac{\partial H'}{\partial x},\ \dot{x}\ =\ \frac{\partial H}{\partial p}\ =\ e^{-2x}\frac{\partial H'}{\partial p},$$

and the dynamical equation includes

$$\dot{p}\ =\ (p^2\ -\ 2e^{4x})e^{2x}\ =\ 2H'e^{-2x},\ \dot{p}'\ =\ -4e^{4x},$$

The reduced Hamiltonian can then be used to describe the dynamics of a lattice or chain of particles. The full Toda lattice Hamiltonian is then

$$H\ =\ \frac{1}{2}\sum_{i=1}^{n}\dot{p}_i^2\ -\ \sum_{i=1}^{n-1}e^{2(x_i\ -\ x_{i-1})}\ +\ e^{2(x_1\ -\ x_n)},$$

where the last term holds if the chain of particles is in a circle.

The standard dynamical variables are $a_i = exp(x_i - x_{i+1})$ and $b_i = p_i$ and the dynamical equations are

$$\dot{a}_i = a_i(b_i - b_{i+1}),\ \dot{b}_j = 2(a^2_{j-1} - a^2_j)$$

This system is equivalent to the a Lax pair system.

The motivation is then to describe a one dimensional chain of atoms according to Newtonian-like potentials in an integrable manner. This is a Kronig-Penny type of model that has soliton wave mechanics.

5 Quantum states and BMS

The commutators for the BMS system above clearly indicates a manner in which the holographic principle can be expressed according to quantum operators. The Quantum Error Correction Code (QECC) is a system that is capable of projecting a set of qubits onto an ancillary set of quantum states that act as a sieve to remove quantum bits outside of the code and to restore erroneous bits. This is a process that involve projector operators, which ostensibly is not unitary. This is then problematic in its internal working. However, if we regard the ancillary set of quantum states are not actually real, then this whole procedure is somewhat fictional. This may then be interpreted as a stand in for a more physically realistic process. This process is related to the physics of fluxes Φ on the horizons r_+ and r_- of a Kerr-Newman black hole

$$\int_\Sigma \Phi = \int_\Sigma^+ \Phi_+ + \int_\Sigma^- \Phi_- = \int_{II} d\Phi. \tag{5.1}$$

that determines a set of states in the volume of the region II of a Kerr-Newman spacetime. The states $\psi = \psi_0 exp(i\int_\Sigma \Phi)$, so that

$$\int_\Sigma \Phi = \int_{II} d\Phi = -iln\left(\frac{\psi}{\psi_0}\right).$$

The functional differential of these fluxes

$$\delta\int_{II} d\Phi = \delta\Phi = -i\delta ln\left(\frac{\psi}{\psi_0}\right) = -i\delta\psi$$

determines the shift in states between the horizon, and this is interpreted as transformations, unitary or otherwise, in these states. The case of unitarity is

$$U(t)|i\rangle|0\rangle = \sum_{n,j}^{i} C^i_{n,j}|j\rangle|n\rangle.$$

For the quantum error correction code we will use the recovery operation, which was developed by Verlinde and Verlinde [11].

Consider a Hilbert space $\mathcal{H}_{an}$ of ancillary states that tensors with the Hilbert space for a black hole $\mathcal{H}_{bh}$ as $\mathcal{H} = \mathcal{H}_{bh} \otimes \mathcal{H}_{an}$. The ancillary states $\mathcal{H}_{an}$ are isomorphic to the space of radiation states $\mathcal{H}_r$. The recovery operator $\mathcal{R}$ that acts on this to produce

$$\mathcal{R}|j\rangle|0\rangle_{an} = \sum_n r_n|j\rangle|n\rangle_{an}.$$

The Hilbert space of ancillary states is where entanglements with the code space and the environment are stored. In this way excess entanglement entropy of the black hole can be swept into this space and removed from the dynamics. This might be objected to as a cheat since while the ancillary Hilbert space might be in some way fictitious, or just a vacuum, storing entanglements in this way makes this state space less than fictional. This space is for now considered to be a gadget to perform algebra, and in the end what ever is in the state space is projected to zero.

Given the unitary evolved state $U|i\rangle|0\rangle_{an}$ the application of $\mathcal{R}$ gives the black hole state $|i\rangle$ multiplied by states in $\mathcal{H}_r \otimes \mathcal{H}_{an}$

$$\mathcal{R}U|i\rangle|0\rangle_{an}|0\rangle_r \;=\; \sum_{m,n}\mathcal{R}_m C_n|i\rangle|m\rangle_{an}|n\rangle_r \;=\; \sum_{m,n,k'} e^{-E_m\beta/2}|k'\rangle\langle k'|C^\dagger_m C_n|i\rangle|m\rangle_{an}|n\rangle_r$$
$$= \sum_{m,n,k'} e^{-E_m\beta/2}e^{E_n\beta}\delta_{mn}|k'\rangle\langle k'|i\rangle|m\rangle_{an}|n\rangle_r \tag{5.2}$$
$$= \sum_{k'}|k'\rangle\langle k'|i\rangle\sum_n e^{E_n\beta/2}|n\rangle_{an}|n\rangle_r \;=\; |i\rangle\sum_n e^{E_n\beta/2}|n\rangle_{an}|n\rangle_r \; if\; |i\rangle \;\in\; \mathcal{H}_{an}$$

This puts the ancillary state in the same eigenstate as the radiation according to the Unruh vacuum $|0\rangle_u \;=\; e^{E_n\beta/2}|n\rangle_r$. The last line obtains if the BH state is in the ancillary states. If not the result is zero.

The Hilbert space of ancillary states is most likely a form of the black hole entanglement Susskind invokes in $ER \;=\; EPR$ key-10b. The ancillary state pertains to the same set of states in $|\{\mathcal{H}_\bigtriangledown$ because in effect they are the same states. This circumvents the violation of quantum monogamy and similarly the firewall problem with black holes.

The details of the operators $C^i_{n,j}$ are not known, but there may be some sort of ergodic matrices formed from an ensemble space of irreducible representations of the E_8. The entropy of these operators is given on a coarse grained level by

$$\sum_j\langle j|C^\dagger_{k,n}C^i_{k,m}|k\rangle \;=\; C^\dagger_n C_m \;\simeq\; e^{-E_n\beta}\delta_{mn},$$

and a finer grained information on these operators would be given by a distribution over the set of irreducible repretatations of the E_8. The operators $C^i_{k,m}$ and $C^{i\dagger}_{k,m}$ are the operator forms of the variables C_{zz} and $C_{\bar z\bar z}$. It is possible to consider the these operators formed according to moments expanded around the black hole states in $\mathcal{H}_{bh}$ according to a Laurent series

$$C^n_a \;=\; \oint\frac{d\xi}{2\pi i}\xi^n C_a(\xi),\; C_a(\xi) \;=\; \sum_{n=-\infty}^{\infty}\xi^{-(n+1)}C^n_a, \tag{5.3}$$

where $\xi \;=\; \bar z \;+\; iz$ and $\bar\xi \;=\; \bar z \;-\; iz$. It is convenient to substitute z for ξ and their conjugates at this point. Similar expansions are possible with the other Hilbert spaces as well.

The primary commutator is

$$[C^m_+,\; C^n_-] \;=\; kn\delta(n \;+\; m) \;-\; 2C^3_{m+n} \tag{5.4}$$

and for $C^3_m \;=\; e^{-i\epsilon_m t}\partial/\partial t$ the commutator of this with $C^\pm_n$ is evaluated on a test function ϕ

$$[C^n_\pm,\; C^m_3]\phi \;=\; C^n_\pm\frac{\partial\phi}{\partial t} \;-\; \frac{\partial}{\partial t}\left(C^n_\pm\phi\right)$$
$$=\; C^n_\pm\frac{\partial\phi}{\partial t} \;-\; e^{-i\epsilon_m t}\left(\frac{\partial C^n_\pm}{\partial t}\phi \;-\; C^n_\pm\frac{\partial\phi}{\partial t}\right) \;=\; e^{-i\epsilon_m t}\frac{\partial C^n_\pm}{\partial t}$$

which we write according to the eigenvalue of this angular momentum operator $C^n_\pm \;=\; e^{\pm i\epsilon t}C^n_\pm|_{t=0}$

$$[C^n_\pm,\; C^m_3]\phi \;=\; \pm\, i\epsilon_n C^{m+n}_\pm. \tag{5.5}$$

In addition we can see the commutator

$$[C_3^m,\ C_3^n]\ =\ An\delta(m+n), \tag{5.6}$$

Equation 18 can be restated with redefinition as the Virasoro form of the $SO(2,\ 1)$ algebra for quasi-spin or boost operators

$$\begin{aligned}[][J_m^+,\ J_n^-]\ &=\ km\delta(m\ +\ n)\ -\ 2J_{m+n}^3\\ [J_m^3,\ J_n^3]\ &=\ -\frac{1}{2}km\delta(m\ +\ n)\\ [J_m^3,\ J_n^3]\ &=\ \pm J_{m+n}^+.\end{aligned} \tag{5.7}$$

A Sugawara coset construction constructs bilinear function of these current operators give moments of the stress-energy tensor

$$T(z)\ \sim\ \frac{1}{k\ -\ 2}g_{ij} : J^i(z)J^j(z) :,$$

to construct operator product expansions of conformal quantum fields [13].

The angular momentum analogue of these operators constructs ladders of states according to states $\ell\rangle$ with

$$J_0^2|\ell\rangle\ =\ \left(J_0^+J_0^-\ -\ J_0^3(J_0^3\ -\ 1)\right)|\ell\rangle$$

and in general

$$J_n^2|\ell\rangle\ =\ \left(J_{-n}^+J_n^-\ -\ J_{-n}^3(J_n^3\ -\ 1)\right)|\ell\rangle$$

and in order to be consistent with the BMS metric we insist on the highest weight representation in $U(1)$ with $\langle\ell|J_n^3J_{-n}^3|\ell\rangle\ =\ 0$. This is a coset construction that eliminates these dynamics along a spacetime direction. This forces this algebra to be consistent with this elementary or reduced form of the BMS metric. This current algebra is then the coset construction $SO(2,\ 1)/U(1)$.

This is a conformal theory in one dimension. For $U(1)\ \to\ SO(1,\ 1)$ or $\mathbb{S}^1\ \to\ \mathbb{R}$ the coset construction is $AdS_2\ \simeq\ SO(2,\ 1)/SO(1,1)$. This map may be realized through the "lifting" on the $U(1)$ bundle. By the Maldecena correspondence this is equivalent to CFT_1.

The extension to $SL(2,\ \mathbb{C})\ \sim\ SO(3,\ 1)$ is the coset construction $SO(3,\ 1)/SU(2)$ with the Wick rotation $SU(2)\ \to\ SU(1,\ 1)\sim\ SO(2,\ 1)$ this is

$$SO(2,\ 2)/SO(2,1)\ =\ AdS_3\ \sim\ CFT_2.$$

This may be continued to $SL(4,\ \mathbb{C})\ \sim\ SO(7,1)$ with the coset $SO(7,\ 1)/SU(4)$. $SU(4)\ \sim\ SO(6)$ the Wick rotation of the divisor $SO(6,2)/SO(5,1)$ is such that the decomposition

$$SO(6,2)/SO(5,1)\ \to\ (SO(6,2)/SO(6,1))\times(SO(6,1)/SO(5,1))\ \sim\ AdS_7\times\mathcal{H}^6$$

results in the AdS_7 tensored with a hyperbolic space $\mathcal{H}^6$. We may think of this as a type of symmetry breaking.

For a group G and a subgroup K that G is spontaneously broken, the broken generators are axial in chiral symmetry greaking in low energy QCD, $SU(2)\times SU(2)/SU(2)_{isospin}$ in the coset space $H\ =\ G/K$. The generators $\mathfrak{g}$ of G split into the unbroken isospin generators $\mathfrak{k}$ and the broken ones $\mathfrak{h}$. The generators $\mathfrak{h}$ of H are parameterized by goldstone bosons or pions as projective coordinates in the manifold S^3 in QCD. In this higher dimensional setting the unbroken symmetries are $SO(5,\ 1)\ \sim\ SU(2,\ 2)$, with 13 generators for the quotient space in the broken symmetry scheme. The further decomposition of the quotient group into $AdS_7\times\mathcal{H}^6$ partitions the 13 generators into these spaces.

The quotient space may be made a Hermitian symmetric space as $BDI_{6,2} = SO(6,2)/SO(5,1)\times U(1)$. The above decomposition in a symmetry breaking then defines the AdS_7 spacetime tensored with the 5 dimensional hyperbolic space $SO(5,1)/SO(4)\times U(1)$. This is then contained in the CFT_6 tensored with a hyperboloid in 6 dimensions. This Hermitian symmetric space is a form of Vafa's F-theory [20] in 12 dimensions. These coset constructions are then conformal or embed conformal fields according to AdS/CFT correspondence. The AdS_7 is equivalent to the $6D-(2,0)$ conformal field theory on the 5 dimensional boundary ∂AdS_7. This has been demonstrated by Witten to give physical justification of the Langland's geometric conjecture [14].

The corresponding 10 dimensional for type IIB strings is the $AdS_5\times S^5$ theory, for the AdS_5 equivalent to $\mathcal{N} = 4$ conformal supersymmetric Yang-Mill field on the 4 dimensional boundary of AdS_5. This by AGT correspondence is dual to the CFT_6 [15]. This may be extended to a Toda lattice theory, which is discussed below.

6 Black holes and cosmologies as scattering amplitudes

The equivalency between the black hole SLOCC and the algebra of the BMS transformation suggests that a black hole can be described according to a scattering process. The BMS supertranslation of quantum information can be defined from $\mathcal{I}^-$ to the black hole horizon and from the black hole horizon to $\mathcal{I}^+$. The equivalency of quantum information with the black hole qubit correspondence and the BMS algebra of supertranslations means the process of black hole formation and evaporation is described by a scattering amplitude. The physics is then entirely unitary or by similar means that conserves quantum information.

The connection to holography can be also be considered with the inverse problem. A black hole that is extremal pushes $r_+ \leftrightarrow r_-$ so the spacelike region disappears. Carroll and Randall demonstrated that this region discontinuously changes into $AdS_2\times S^2$. For a 10 dimensional black hole this interior region is transformed discontinuously into $AdS_5\times S^5$. We may think of the standard 4 dimensional black hole as possessing a Calabi-Yau compactified space that when uncompactified is $AdS_5\times S^5$. This means there is a reciprocal process whereby a black hole near the extremal limit fluctuates into the extremal condition momentarily and then returns to the near extremal condition. This quantum tunneling results in the generation of a cosmology that projects into an additional time direction.

Much of the mathematical basis for this involves the many body problem. Newtonian mechanics for dynamics in a central gravity field is the Hamiltonian $H = \frac{1}{2m}p^2 + kr^{-2}$. For $u = 1/r$ and further with $x = ln(u)$ the Hamiltonian for $p_x = m\dot x$ is

$$H = \frac{1}{2m}\sum_i \left(p_{x_i}^2 + 4ke^{2x_i}\right).$$

The Toda lattice then considers exact solutions of many body systems in spatial orientations given by the root vectors of Lie algebras. In the Lax pair representation with equation 12 this Hamiltonian above becomes equation 14.

Physical connection to holography can be seen according to string theory. Open strings with open ends may be thought of as a tethered system of two masses. The Lorentz boosted string near the black hole horizon spreads out across the horizon as seen by a distant observer. The string merges with other strings on the horizon. At the stretched horizon, we may think of the black holes as determined by one large string that winds through every Planck unit of horizon area. This one "mega-string" that is a form of loaded string. We may think of this string as one huge lattice of strings, where they are linked together as if they had masses on their connecting endpoints. We may think of the endpoints sitting on each Planck unit of area on the horizon. These endpoints are associated with Chan-Paton factors. The Wilson line the string has a holonomy, In n dimensions the pure gauge

$$\chi \;=\; \frac{X^n}{2\pi R} diag(x_i,\; x_2,\; \dots,\; x_n)$$

is local, with the Wilson integral $W \;=\; \int d\chi$ periodic. This holonomy is removed so the group, such as $SL(N,\; \mathbb{C})$, picks up a phase. In a lattice system we may then have the algebra g of the group G decomposed as $g \;=\; h \;+\; k$, so the quotient $H \;=\; G/K$ removes this holonomy. This is a version of the standard approach that assigns a phase to states $|ij\rangle$ charged in $SL(N,\; \mathbb{C})$ so that

$$p^n \;=\; \frac{m}{R} \;+\; \frac{x_i \;-\; x_j}{2\pi R}$$

and the phase $exp(i(x_i \;-\; x_j))$ cancels the holonomy. This is a form of quotient group construction. In similar manner strings on the horizon of a black hole are a linear chain with unitary coset group construction. The Chan-Paton degree of freedom is then a reduced system that describes the motion of the string endpoint on a space.

The elementary string Hamiltonian

$$H \;=\; \frac{1}{2}\left[\left(\frac{\partial X}{\partial \tau}\right)^2 \;+\; \left(\frac{\partial X}{\partial \sigma}\right)^2\right]$$

for $X \;=\; e^q$ is

$$H \;=\; \frac{1}{2}\left[\left(\frac{\partial q}{\partial \tau}\right)^2 \;+\; \left(\frac{\partial q}{\partial \sigma}\right)^2\right] e^{2q}.$$

If we let $(\partial q/\partial \tau)e^q \;=\; p$ and $\partial q/\partial \sigma \;=\; k$ the Hamiltonian is the

$$H \;=\; \frac{1}{2}\left(p^2 \;+\; k^2 e^{2q}\right).$$

A chain of these string leads naturally to the Toda lattice Hamiltonian

$$H \;=\; \frac{1}{2}\left(\sum_{i=1}^{n} p_n^2 \;+\; k^2 \sum_{i=1}^{n-1} e^{2(q_{i+1} \;-\; q_i)}\right) \;+\; e^{2(q_1 \;-\; q_n)} \tag{6.1}$$

where the last term is included for a closed periodic chain, and dropped if the chain is open. For $a_i \;=\; p_i$ and $b_i \;=\; exp(q_{i+1} \;-\; q_i)$ the dynamics is

$$\dot{a}_i \;=\; a_i(b_{i+1} \;-\; b_i),\; \dot{b}_i \;=\; 2(a_i^2 \;-\; a_{i-1}^2). \tag{6.2}$$

For an n chained lattice there is then ADE algebraic systems $A_n,\; B_n, D_n$ or for $n \;=\; 2,\; 4$ the groups G_2, F_4 and for $6,\; 7,\; 8$ the E_n groups.

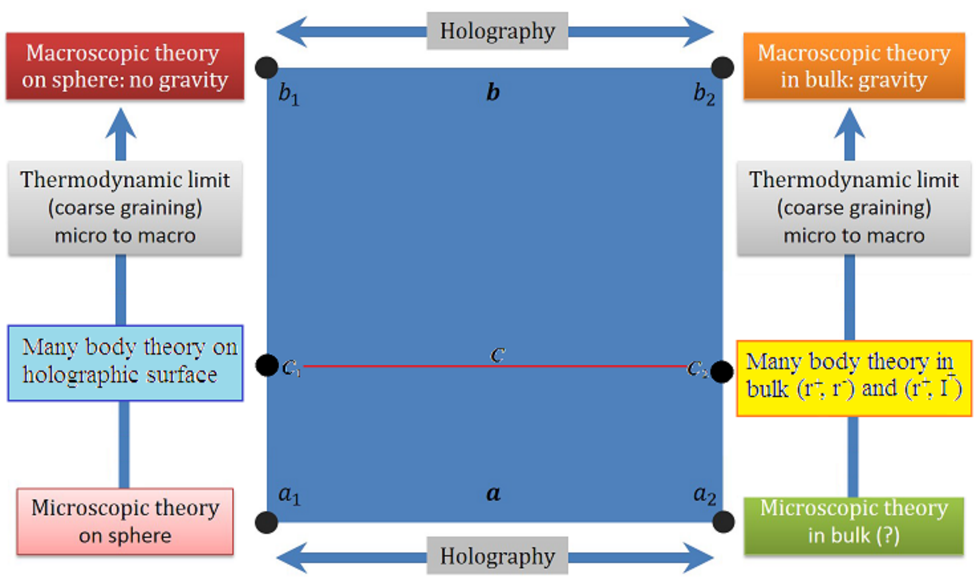
The Susskind grand string that covers the horizon of a black hole is then an ensemble of strings with n endpoints connected as a loaded string. The Bott periodicity with Lie groups also indicates there is a topological cyclicity of fields [16]. By Bott periodicity the topological properties of all $SU(2n)$ groups are equivalent. The $SU(2) \;\simeq\; SO(3)$, which means the topological properties of these fields on this surface is the $2 \;+\; 1$ spin Hall effect. The stretched horizon is a tangle of loops and handles, which determine this large N expansion. However, the primary topological field properties are those of the smallest group $SU(2)$. The SPT states apply for large N, and the stretched horizon exists in a different phase. This topological phase is additional hair or fuzz on the stretched horizon. This hair is proposed by Mathur as a resolution to the firewall problem[17]. A similar idea based on BMS symmetry has been advanced by Hawking, Perry and Strominger[18]

A phase change in quantum states near the horizon permits states to spread into a set of SPT states which comprise fuzzball states. The quantum critical, or analogous thermal induced, phase occur from two perspectives. The accelerated or asymptotic observer witnesses quantum states on the horizon time dilated and red shifted so UV states and their extreme phase is observable. The thermal analogue would be seen by an accelerated observer near the horizon. This observer sees the rapid passage of events and a rapid flux of radiation in and out of the black hole. The hot Hawking radiation would exhibit a thermal induced phase approaching the Hagedorn temperature, where a phase change occurs. The quantum critical and thermal induced phases are complements of each other. The occurrence of states in this SPT phase provides the fuzzball set of states. This provides an avenue for the resolution of the AMPs firewall problem[13].

The density of states of the black hole is given by the integer partition function. The density of states for strings has a generating function that in an asymptotic limit is given by the Hardy-Ramanujan formula. This formula is an approximation to the integer partition. The mass of a black hole is its energy and the number of Planck areas on the horizon means that $E_n ~=~ c\sqrt{n}$ and the partition function for a Schwarzschild black hole is

$$Z(\beta) ~=~ \sum_E g(E)e^{-\beta E} \tag{6.3}$$

for a counting of states leads to a form of the Hardy-Ramanujan formula, similar to the analysis in [19]. The counting of irreducible representations with Youngs Tableaux of the large N expansion of a YM field on the horizon leads to the integer partition.



It is remarkable that an aspect of Newtonian classical mechanics, in effect the zero energy limit of quantum gravity, even lower than the low energy limit of general relativity, contains structure that pertains to a high energy limit with quantum gravity and holography. This appearance with respect to high energy physics is seen in the diagram modified from [21]. The many body physics here is a closer approximation to reality than $N ~\rightarrow~ \infty$ physics analogous to statistical mechanics. This is the high energy limit. The converse low energy limit is with respect to exact solutions of Newtonian mechanics that are not of great practical value to the astro-dynamicists, though these can form pertubation terms. For instance consider the $A_2 = SU(3)$ for three bodies. The $\mathbf{3}\times\bar{\mathbf{3}}$ root system of a triangle is an elementary model of three equal mass stars in a mutual isosceles triangle in rotation. A slight perturbation of one of the stars can be thought of as a series with A_3 for a fourth satellite in motion. These exact solutions are then islands of rationality with some domain of approximation around them in a solution space that is chaotic dynamics.

This then points to an aspect of renormalization group flow between high energy physics and low energy physics. In this case it is how quantum gravity has as its low energy target Newtonian mechanics and Kepler's laws. In fact the Kepler's second law of equal area equal time and his third law on the periodicity square proportional to the cube of the orbital radius is suggestive of the holographic principle in one and two dimensions and two and three dimensions. This then connects physics with Yang-Mills gauge fields and gravity with questions concerned with the quantum-classical correspondence.

It is interesting to speculate what happens with the unknown microscopic theory in the bulk. It might be that this is related to the chaos theory, and potentially quantum chaos. The reduction in N to a unit or zero (the vacuum) many then correspond to Hamiltonian chaos theory and potentially quantum chaos.

Received December 1, 2016; Accepted December 25, 2016

References

[1] H. Bondi, M. G. J. van der Burg, A. W. K. Metzner, "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems," *The Royal Society Proceedings A* **269**, 1336 (1962) http://rspa.royalsocietypublishing.org/content/269/1336/21

[2] L. Borsten, M.J. Duff, A. Marrani, W. Rubens, "On the Black-Hole/Qubit Correspondence," *Eur. Phys. J. Plus* **126**, 37 (2011) http://arxiv.org/abs/1101.3559v2

[3] M. Favata "The gravitational-wave memory effect," *Class. Quant. Grav.* **27**, 084036, (2010) http://arxiv.org/abs/1003.3486

[4] V. B. Braginsky, K. S. Thorne, "Gravitational Wave Bursts with Memory and Experimental Prospects," *Nature*, **327**, 123-125 (1987).

[5] A. Jamiolkowski, *Rep. Math. Phys.* **3**, 275 (1972)

[6] M. S. Leifer, R. W. Spekkens, "Towards a Formulation of Quantum Theory as a Causally Neutral Theory of Bayesian Inference," *Phys. Rev. A* **88**, 052130 (2013)

[7] A. Strominger and A. Zhiboedov, "Gravitational Memory, BMS Supertranslations and Soft Theorems," http://arxiv.org/abs/1411.5745

[8] T. He, V. Lysov, P. Mitra and A. Strominger, "BMS Supertranslations and Weinbergs Soft Graviton Theorem," http://arxiv.org/abs/1401.7026v2

[9] M. A. Olseshenetsky. A. M. Perelomov, Explicit solutions of the Calogero model in the classical case and geodesic flows on symmetric spaces of zero curvature, *Lett.NuovoCim.* **16**, 333-339 (1976)

[10] T. He, V. Lysov, P. Mitra and Andrew Strominger, "BMS Supertranslations and Weinbergs Soft Graviton Theorem," http://arxiv.org/abs/1401.7026

[11] E. Verlinde, H. Verlinde, "Black Hole Entanglement and Quantum Error Correction," JHEP **10** 107 (2013) http://arxiv.org/abs/1211.6913

[12] L. Susskind, "ER=EPR, GHZ, and the Consistency of Quantum Measurements," http://arxiv.org/abs/1412.8483

[13] L. J. Dixon, M. E. Peskin, $N = 2$ Superconformal Symmetry and $SO(2, 1)$ Current Algebra, Nucl. Phys. **B325** 349-355 (1989)

[14] E. Witten, "Geometric Langlands from six dimensions". (2009). arXiv:0905.2720

[15] N. Wyllard, "A(N-1) conformal Toda field theory correlation functions from conformal $N = 2$ $SU(N)$ quiver gauge theories". Journal of High Energy Physics. **11** 002 (2009). arXiv:0907.2189

[16] L. B. Crowell, "Topology of States on a Black Hole Event Horizon," EJTP **12**, IYL15-34 211-218 (2015)

[17] S. D. Mathur, "What does strong subadditivity tell us about black holes?," http://arxiv.org/abs/1309.6583

[18] S. W. Hawking, M. J. Perry, A. Strominger,"Soft Hair on Black Holes," Phys. Rev. Lett. **116**, 231301 (2016)

[19] M. B. Green, J. H. Schwarz, E. Witten, Superstring Theory, Cambridge (1984) pp 116-119

[20] C. Vafa. "Evidence for F-theory." Nucl.Phys. **B469**: 403-418, (1996 DOI:10.1016/0550-3213(96)00172-1. arXiv: hep-th/9602022

[21] D. Dieks, J. van Dongen1 and S. de Haro, Emergence in Holographic Scenarios for Gravity ($http://philsci-archive.pitt.edu/11669/4/Emergence\_hol\_gravity\_July\_2015.pdf$)