Article

Cohen's Formula Applied to Laplace-Chebyshev-Hermite Polynomials

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Abstract

We employ the Cohen's expansion to obtain the explicit formula and the recurrence relation for the Laplace-Chebyshev-Hermite polynomials.

Keywords: Cohen's formula, Laplace-Chebyshev-Hermite, polynomials, Expansion of operators.

1. Introduction

Cohen [1, 2] proved the expansion:

$$(\lambda\, x+p)^n=\sum_{k=0}^{[\frac{n}{2}]}\sum_{r=0}^{n-2k}\frac{(-1)^k\, n!}{k!\,(n-2k)!}\binom{n-2k}{r}\left(\frac{i\hbar}{2}\right)^k\lambda^{n-k-r}\,x^{n-2k-r}\,p^r\,, \tag{1}$$

such that all x factors are to the left of the p factors, and:

$$p=-i\hbar\,\frac{d}{dx}, \qquad [x,p]=i\hbar\,. \tag{2}$$

If we use $\lambda=-\varepsilon\, i\hbar$, then (1) acquires the form:

$$\left(\varepsilon\, x+\frac{d}{dx}\right)^n=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{1}{k!\,2^k}\sum_{r=0}^{n-2k}\frac{\varepsilon^{n-k-r}}{r!\,(n-2k-r)!}\,x^{n-2k-r}\,\frac{d^r}{dx^r}\,, \tag{3}$$

where ε is an arbitrary constant.

In Sec. 2 we apply (3) to the Laplace [3]-Chebyshev [4]-Hermite [5] polynomials $H_m(x)$ to deduce the corresponding recurrence relation and the explicit formula.

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2. Laplace-Chebyshev-Hermite polynomials

The polynomials under study verify the differential equation [6, 7]:

$$H_n'' - 2\,x\,H_n' + 2\,n\,H_n = 0, \qquad n = 0, 1, 2, \dots \tag{4}$$

that is:

$$H_0 = 1, \qquad H_1 = 2\,x, \qquad H_2 = 4\,x^2 - 2, \qquad H_3 = 8\,x^3 - 12\,x, \ \dots \tag{5}$$

in according with the following annihilation and creation operators [8]:

$$H_n = \left(2\,x - \frac{d}{dx}\right) H_{n-1} = (2\,x - \frac{d}{dx})^n\, H_0,$$
$$\frac{d}{dx}\,H_n = 2\,n\,H_{n-1} \qquad \therefore \qquad \frac{d^k}{dx^k}\,H_n = 2^k\,\frac{n!}{(n-k)!}\,H_{n-k}\,. \tag{6}$$

On the other hand, for $\varepsilon = -2$ the expression (3) gives:

$$(2\,x - \frac{d}{dx})^n = n!\,\sum_{k=0}^{[\frac{n}{2}]}\sum_{r=0}^{n-2k}\frac{(-1)^{k+r}}{k!\,r!\,(n-2k-r)!}\,(2\,x)^{n-2k-r}\,\frac{d^r}{dx^r}\,, \tag{7}$$

whose application to H_0 implies the known explicit formula [6]:

$$H_n(x) = n!\;2^n\,\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k}{k!\,(n-2k)!\,4^k}\,x^{n-2k}\,. \tag{8}$$

Besides, the action of (7) on H_m leads to:

$$H_{m+n}(x) = m!\,n!\;2^n\sum_{k=0}^{[\frac{n}{2}]}\frac{1}{k!\,4^k}\sum_{r=0}^{n-2k}\frac{(-1)^{k+r}}{r!\,(m-r)!\,(n-2k-r)!}\,x^{n-2k-r}\,H_{m-r}(x), \tag{9}$$

and for the special case $n = 1$ we obtain the recurrence relation [6]:

$$H_{m+1} - 2\,x\,H_m + \,2\,m\,H_{m-1} = 0, \qquad m = 1, 2, \dots \tag{10}$$

Thus we see that the Cohen's expansion [1, 2] is useful to deduce properties of the important Laplace-Chebyshev-Hermite polynomials.

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