Article

Wave Equation for the Lanczos Spintensor

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Abstract

We exhibit the wave equation satisfied by the Lanczos potential in arbitrary spacetimes.

Keywords: Weyl tensor, Lanczos generator, wave equation.

1. Introduction

Here we employ the notation and quantities explained in [1, 2]. In any $R_4$ the Lanczos potential [3] $K_{\mu\nu\alpha}$ generates to the conformal tensor via the relation:

\[ C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} + K_{\mu\beta\nu;\alpha} - K_{\mu\alpha\nu} g_{\beta\mu} + K_{\nu\alpha\beta} g_{\mu\beta} - K_{\nu\beta\mu} g_{\alpha\mu}, \]

(1)

with the properties:

\[ K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \quad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0, \quad K_{\mu\nu} \equiv K_{\mu\alpha\nu}^{;\alpha} = K_{\nu\mu}, \]

(2)

$K^\mu_{\nu;\nu} = 0$ Algebraic Lanczos gauge, $K^{\mu\nu;\alpha} = 0$ Differential Lanczos gauge.

The Sec. 2 contains tensor and spinor expressions for $\Box K_{\mu\nu\alpha}$, where $\Box = \nabla^\beta \nabla_\beta$ is the differential wave operator in spacetime, in accordance with [4-7].

2. Wave equation for the Lanczos potential

In (1) we apply $;^{\mu}$, the non-commutative property of the covariant derivative [8], and (2) to obtain the wave equation [5-7]:

\[ \Box K_{\alpha\beta\nu} = C_{\alpha\beta\nu;\mu} + 2 \left( R_{\mu\nu} K_{\alpha\beta}^{;\mu} + R_{\mu\lambda} K^{\mu}_{\lambda}[\alpha \ g_{\beta\nu}] - R^{\mu}_{\mu}[\alpha K_{\beta\nu}] \right) - \frac{1}{2} R K_{\alpha\beta\nu}, \]

(3)
where \( R_{\mu\nu} \equiv R^\lambda_{\mu\nu\lambda} \) and \( R \equiv R^\mu_{\mu} \) are the Ricci tensor and the scalar curvature, respectively. The Bianchi identities [8] participate in (3):

\[
C_{\alpha\beta\nu\mu} \propto = -J_{\nu[\alpha;\beta]}, \quad J_{\nu\lambda} \equiv R_{\nu\lambda} - \frac{1}{6} R g_{\nu\lambda};
\]

(4)

hence in vacuum \((R_{\mu\nu} = 0)\), from (3) it is clear that the Lanczos potential satisfies the wave equation \( \Box K_{\mu\nu} = 0 \), as occurs in Kerr metric [9, 10].

In the deduction of (3) it is important to know the interesting identity valid only in four dimensions [6, 7]:

\[
K^\rho_\mu K_{\mu\lambda}[\alpha g_{\beta\nu}] + 2 C_{\mu\nu\mu[K}\rho_{\beta\mu]}^\rho + \frac{1}{2} K^\mu_{\mu\nu} C_{\mu\rho\alpha\beta} = 0,
\]

(5)

which can be proved employing the generalized Kronecker delta [11].

With the spinor analysis exposed in [1, 2, 12] we can construct the spinorial version of (3) [4, 7, 13]:

\[
L_{A\cap B\cap C\cap D} \equiv -\nabla_E^D \psi_{ABCE} - 6 \phi^E_{(A} \phi_{BC)EF} \frac{1}{4} R L_{ABC\cap D} \epsilon,
\]

(6)

where \( L_{A\cap B\cap C\cap D} \) is the Lanczos spinor, and the Bianchi identities take the form [12]:

\[
\nabla_E^D \psi_{ABCE} = -\nabla_{(A} \phi_{BC)F} \phi_{\cap D}, \quad \nabla^{BC \cap D} \phi_{AB\cap CD} = -\frac{1}{8} \nabla_{AD} R,
\]

(7)

in terms of the Weyl and matter spinors.

Finally, we indicate that the spinor transcription of the Weyl-Lanczos equations (1) with both gauge conditions (2) imposed is given by:

\[
\psi_{ABCD} = 2 \nabla_D \phi_{ABCE}.
\]

(8)

The relations (3), (5), (6) and (8) can be extended to the case \( K_{\mu\nu\alpha} \neq 0 \) [4, 7].

**References**


