Article

Wave Equation for the Lanczos Spintensor

S. Álvarez-Ballesteros, R. Cruz-Santiago & J. López-Bonilla*

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Lindavista 07738, CDMX, México

Abstract

We exhibit the wave equation satisfied by the Lanczos potential in arbitrary spacetimes.

Keywords: Weyl tensor, Lanczos generator, wave equation.

1. Introduction

Here we employ the notation and quantities explained in [1, 2]. In any R_4 the Lanczos potential [3] $K_{\mu\nu\alpha}$ generates to the conformal tensor via the relation:

$$C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\nu;\mu} + K_{\mu\beta}\, g_{\nu\alpha} - K_{\mu\alpha}\, g_{\nu\beta} + K_{\nu\alpha}\, g_{\mu\beta} - K_{\nu\beta}\, g_{\mu\alpha}, \qquad (1)$$

with the properties:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}\,, \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0, \qquad K_{\mu\nu} \equiv K_{\mu\alpha\nu}{}^{;\alpha} = K_{\nu\mu}\,, \qquad (2)$$

$K^{\mu\nu}{}_{\nu} = 0$ Algebraic Lanczos gauge, $K^{\mu\nu\alpha}{}_{;\alpha} = 0$ Differential Lanczos gauge.

The Sec. 2 contains tensor and spinor expressions for $\Box K_{\mu\nu\alpha}$, where $\Box = \nabla^{\beta}\,\nabla_{\beta}$ is the differential wave operator in spacetime, in accordance with [4-7].

2. Wave equation for the Lanczos potential

In (1) we apply $^{;\mu}$, the non-commutative property of the covariant derivative [8], and (2) to obtain the wave equation [5-7]:

$$\Box K_{\alpha\beta\nu} = C_{\alpha\beta\nu\mu}{}^{;\mu} + 2\left(R_{\mu\nu}\, K_{\alpha\beta}{}^{\mu} + R_{\mu\lambda}\, K^{\mu}{}_{[\alpha}{}^{\lambda}\, g_{\beta]\nu} - R^{\mu}{}_{[\alpha}\, K_{\beta]\nu\mu}\right) - \frac{1}{2} R\, K_{\alpha\beta\nu}\,, \qquad (3)$$

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, CDMX, México
E-mail: jlopezb@ipn.mx

where $R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\nu\lambda}$ and $R \equiv R^{\mu}{}_{\mu}$ are the Ricci tensor and the scalar curvature, respectively. The Bianchi identities [8] participate in (3):

$$C_{\alpha\beta\nu\mu}{}^{;\mu} = -J_{\nu[\alpha;\beta],} \qquad J_{\nu\lambda} \equiv R_{\nu\lambda} - \frac{1}{6} R\, g_{\nu\lambda}\,; \tag{4}$$

hence in vacuum ($R_{\mu\nu} = 0$), from (3) it is clear that the Lanczos potential satisfies the wave equation $\Box K_{\alpha\beta\nu} = 0$, as occurs in Kerr metric [9, 10].

In the deduction of (3) it is important to know the interesting identity valid only in four dimensions [6, 7]:

$$K^{\mu\rho\lambda}\, C_{\mu\rho\lambda[\alpha}\, g_{\beta]\nu} + 2\, C_{\mu\nu\rho[\alpha}\, K_{\beta]}{}^{\mu\rho} + \frac{1}{2} K^{\mu\rho}{}_{\nu}\, C_{\mu\rho\alpha\beta} = 0, \tag{5}$$

which can be proved employing the generalized Kronecker delta [11].

With the spinor analysis exposed in [1, 2, 12] we can to construct the spinorial version of (3) [4, 7, 13]:

$$L_{ABC}{}^{\dot{D}} = -\nabla^{E\dot{D}} \psi_{ABCE} - 6\, \phi^{E}{}_{(A}{}^{\dot{F}\dot{D}}\, L_{BC)E\dot{F}} + \frac{1}{4} R L_{ABC}{}^{\dot{D}}\,, \tag{6}$$

where $L_{ABC\dot{D}}$ is the Lanczos spinor, and the Bianchi identities take the form [12]:

$$\nabla^{E}{}_{\dot{D}}\, \psi_{ABCE} = -\nabla_{(\mathrm{A}}{}^{\dot{F}}\, \phi_{BC)\dot{F}\dot{D}}\,, \qquad \nabla^{B\dot{C}} \phi_{AB\dot{C}\dot{D}} = -\frac{1}{8} \nabla_{A\dot{D}}\, R\,, \tag{7}$$

in terms of the Weyl and matter spinors.

Finally, we indicate that the spinor transcription of the Weyl-Lanczos equations (1) with both gauge conditions (2) imposed is given by:

$$\psi_{ABCD} = 2\, \nabla_{D}{}^{\dot{E}} L_{ABC\dot{E}}\,. \tag{8}$$

The relations (3), (5), (6) and (8) can be extended to the case $K_{\mu\nu\alpha}{}^{;\alpha} \neq 0$ [4, 7].

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