Article

Bianchi Type-III Cosmological Model in $f(R)$ Theory of Gravity

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Abstract

In this paper, we study the isotropic perfect fluid solutions of Bianchi Type-III space time in the metric version of *f(R)* gravity. The field equations are solved by taking expansion scalar θ proportional to shear scalar σ which gives $C = A^n$, where A and C are the metric coefficients. The physical behavior of the solutions has been discussed using some physical quantities. Also, the function *f(R)* of the Ricci scalar is evaluated.

Keywords: *f(R)* gravity, Bianchi, Type-III, perfect fluid.

1. Introduction

The astrophysical data show that the current universe passes through a phase of accelerating expansion [1-4]. This phase is driven by a kind of unknown component, dubbed dark energy [5, 6]. In physical cosmology, the simplest candidate for dark energy is the cosmological constant (Λ) [7] satisfying the equation of state $\omega = -1$. But it is plagued with the fine-tuning and the cosmic coincidence problems. To solve these problems, a variable Λ was introduced such that Λ was large in the early universe and then decayed with evolution [8]. Cosmological scenarios with a time-varying Λ [9-13] were investigated by several researchers. On the otherhand, a variety of scalar field models have been proposed in the literature such as quintessence [14-16], k-essence [17,18], tachyon [19], dilation [20], phantom dark energy [21] etc.

Recently, it has been found that many dark energy models get into trouble when tested by some old red-shift objects [22, 23]. In order to understand the problem of accelerating expansion, the study of modified gravity theories also has become very popular. There are several modified gravity theories such as *f(R)* gravity [24, 25], *f(T)* gravity [26], *f(R,T)* gravity [27, 28], *f(G)* gravity [29] etc.

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Among the various modified gravity theories, the *f(R)* theory of gravity is treated most seriously during the last decade. These models consist of higher order curvature invariants as functions of the Ricci scalar. Viable *f(R)* gravity models [30] have been proposed which show the unification of early time inflation and late time acceleration. The problem of dark matter can also be addressed by using viable *f(R)* gravity models. Lobo and Oliveira [31] constructed wormhole geometries in the context of *f(R)* theory of gravity. Multamaki and Vilja [32] investigated spherically symmetric vacuum solutions and later [33] they studied the perfect fluid solutions in *f(R)* gravity. Azadi et al. [34] studied cylindrically symmetric vacuum solutions in this theory. Paul et al. [35] studied Friedmann- Robertson- Walker cosmologies in *f(R)* gravity. Sharif and Shamir [36, 37] investigated the solutions of Bianchi types I and V space times in the framework of *f(R)* gravity.

The purpose of present work is to investigate the perfect fluid solutions of Bianchi type III spacetime in metric *f(R)* gravity. The paper is organized as follows: in Sec. 2, we briefly give the field equations in metric *f(R)* gravity. In Sec.3, we have solved the Bianchi type III space times in *f(R)* gravity and in the last section we discuss the results.

2. *f*(*R*) gravity formalism

The action for *f(R)* gravity is given by

$$S = \int \sqrt{-g}\left[\frac{1}{2\kappa}f(R) + L_m\right]d^4x \tag{1}$$

where *f(R)* is a general function of the Ricci scalar, $\kappa = 8\pi G = 1$, L_m is the matter Lagrangian and g is the determinant of the metric tensor g_{ij}.

Taking variation of the action (1) with respect to the metric tensor g_{ij} yields field equation as

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i\nabla_j F(R) + g_{ij}\Box F(R) = T_{ij} \tag{2}$$

where $F(R) = \frac{df(R)}{dR}$, $\Box \equiv \nabla^i\nabla_i$, ∇_i is the covariant derivative and T_{ij} is the standard matter energy-momentum tensor derived from the Lagrangian L_m.

3. Field equations and solutions

The spatially homogeneous and anisotropic Bianchi type III space time is given by

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{-2\alpha x}dy^2 - C^2(t)dz^2 \tag{3}$$

where A, B, C are scale factors and α is a constant.

The corresponding Ricci scalar curvature is given by

$$R = -2\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{C}}{C}\frac{\dot{A}}{A}\right] \tag{4}$$

The energy momentum tensor for perfect fluid is given by

$$T_{ij} = diag[\rho, -p, -p, -p] \tag{5}$$

where ρ and p are density and pressure respectively.

The field equations (2) with (5) for the metric (3) subsequently lead to the following system of equations

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right)F + \frac{1}{2}f(R) - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{F} = -\rho \tag{6}$$

$$\left(\frac{\ddot{A}}{A} - \frac{\alpha^2}{A^2} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\frac{\dot{A}}{A}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{F} = p \tag{7}$$

$$\left(\frac{\ddot{B}}{B} - \frac{\alpha^2}{A^2} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\dot{F} = p \tag{8}$$

$$\left(\frac{\ddot{C}}{C} + \frac{\dot{C}}{C}\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\frac{\dot{C}}{C}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{F} = p \tag{9}$$

$$\alpha\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)F = 0 \tag{10}$$

From equation (10) we get

$$A = c_1 B \tag{11}$$

where c_1 is the constant of integration. Without any loss of generality, we take $c_1 = 1$ for the sake of simplicity. Using this value of *B* in the above equations, we obtain

$$\left(2\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C}\right)F + \frac{1}{2}f(R) - \left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\dot{F} = -\rho \tag{12}$$

$$\left(\frac{\ddot{A}}{A} - \frac{\alpha^2}{A^2} + \frac{\dot{A}^2}{A^2} + \frac{\dot{A}}{A}\frac{\dot{C}}{C}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\dot{F} = p \tag{13}$$

$$\left(\frac{\ddot{C}}{C} + 2\frac{\dot{C}}{C}\frac{\dot{A}}{A}\right)F + \frac{1}{2}f(R) - \ddot{F} - 2\frac{\dot{A}}{A}\dot{F} = p \tag{14}$$

Subtracting equation (12) from (13) and (14) separately and then dividing by *F*, we get

$$-\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} + \frac{\dot{A}^2}{A^2} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{F}}{F} - \frac{\ddot{F}}{F} = \frac{p+\rho}{F} \tag{15}$$

$$-2\frac{\ddot{A}}{A}+2\frac{\dot{A}}{A}\frac{\dot{C}}{C}+\frac{\dot{C}}{C}\frac{\dot{F}}{F}-\frac{\ddot{F}}{F}=\frac{p+\rho}{F} \tag{16}$$

Now we give definition of some physical quantities before solving these equations.

The average scale factor a and the volume scale factor V are defined as

$$a=\sqrt[3]{A^2Ce^{-\alpha x}}\ ,\ V=a^3=A^2Ce^{-\alpha x} \tag{17}$$

The average Hubble parameter H, expansion scalar θ and shear scalar σ are given in the form

$$H=\frac{\dot{a}}{a}=\frac{1}{3}\left(2\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right) \tag{18}$$

$$\theta=3H=2\frac{\dot{A}}{A}+\frac{\dot{C}}{C} \tag{19}$$

$$\sigma^2=\frac{1}{3}\left[\frac{\dot{A}^2}{A^2}+\frac{\dot{C}^2}{C^2}-2\frac{\dot{A}}{A}\frac{\dot{C}}{C}\right] \tag{20}$$

To solve the equations (15) and (16), we need some additional constraints. One of the additional constraint, we use that the expansion scalar θ is proportional to shear scalar σ which gives

$$C=A^n \tag{21}$$

where $n\neq 1$ is a positive constant.

Using this condition in equations (15), (16) and then subtracting, we obtain

$$\frac{\ddot{A}}{A}+(1+n)\frac{\dot{A}^2}{A^2}-\frac{1}{(1-n)}\frac{\alpha^2}{A^2}+\frac{\dot{A}}{A}\frac{\dot{F}}{F}=0 \tag{22}$$

Now we use the power law relation between F and a [36]

$$F=la^m \tag{23}$$

where l is the constant of proportionality and m is any real number. From equations (18), (21), equation (23) becomes

$$F=lA^{\frac{m}{3}(2+n)} \tag{24}$$

Inserting this value of F in equation (22), it follows that

$$\frac{\ddot{A}}{A}+\frac{3(1+n)+m(2+n)}{3}\frac{\dot{A}^2}{A^2}-\frac{\alpha^2}{1-n}\frac{1}{A^2}=0 \tag{25}$$

Integrating this equation, we obtain

$$\dot{A}=\sqrt{\frac{3\alpha^2}{(1-n)(6+6n+4m+2mn)}+\frac{c_2}{A^{\frac{6+6n+4m+2mn}{3}}}} \tag{26}$$

where c_2 is the integration constant. With the help of (11), (21) the line element (3) reduces to

$$ds^2 = \left[\frac{(1-n)(6+6n+4m+2mn)A^{\frac{6+6n+4m+2mn}{3}}}{3\alpha^2 A^{\frac{6+6n+4m+2mn}{3}}+c_2(1-n)(6+6n+4m+2mn)}\right] dA^2 - A^2dx^2 - A^2e^{-2\alpha x}dy^2 - A^{2n}dz^2 \quad (27)$$

and it takes the form

$$ds^2 = \left[\frac{(1-n)(6+6n+4m+2mn)T^{\frac{6+6n+4m+2mn}{3}}}{3\alpha^2 T^{\frac{6+6n+4m+2mn}{3}}+c_2(1-n)(6+6n+4m+2mn)}\right] dT^2 - T^2dx^2 - T^2e^{-2\alpha x}dy^2 - T^{2n}dz^2 \quad (28)$$

where $A = T$

The expressions for the kinematical parameters i.e. the average Hubble parameter (*H*), expansion scalar (θ) and shear scalar (σ) are obtained as

$$H = \frac{1}{3}(2+n)\left[\frac{3\alpha}{(1-n)(6+6n+4m+2mn)T^2} + \frac{c_2}{T^{\frac{12+6n+4m+2mn}{3}}}\right]^{\frac{1}{2}} \quad (29)$$

$$\theta = (2+n)\left[\frac{3\alpha}{(1-n)(6+6n+4m+2mn)T^2} + \frac{c_2}{T^{\frac{12+6n+4m+2mn}{3}}}\right]^{\frac{1}{2}} \quad (30)$$

$$\sigma^2 = \frac{1}{3}(n-1)^2\left[\frac{3\alpha}{(1-n)(6+6n+4m+2mn)T^2} + \frac{c_2}{T^{\frac{12+6n+4m+2mn}{3}}}\right] \quad (31)$$

The energy density , the scalar curvature and *f(R)* function are obtained as

$$\rho = \frac{lT^{\frac{m}{3}(2+n)-2}}{2(1+\omega)}\left[c_2 T^{-\frac{6+6n+4m+2mn}{3}}\left\{4+8n+\frac{16m}{3}+\frac{16mn}{3}+\frac{4mn^2}{3}\right\} + \frac{3\alpha^2}{(1-n)(6+6n+4m+2mn)}\left\{-1+4n+n^2+\frac{2m}{3}+\frac{7mn}{3}+mn^2-\frac{8m^2}{9}-\frac{8m^2n}{9}-\frac{2m^2n^2}{9}\right\}\right] \quad (32)$$

$$R = \frac{2}{T^2}\left[c_2\left\{\frac{3+6n+4m+4mn+mn^2}{3}\right\}T^{-\frac{6+6n+4m+2mn}{3}} + \alpha^2\left\{\frac{9n^2+3n-3-4m+2mn+2mn^2}{(n-1)(6+6n+4m+2mn)}\right\}\right] \quad (33)$$

$$f(R) = lT^{\frac{m}{3}(2+n)-2}\left[c_2T^{-\frac{6+6n+4m+2mn}{3}}\left\{-(4+8n)-\frac{1}{1+\omega}\left(4+8n+\frac{16m}{3}+\frac{16mn}{3}+\frac{4mn^2}{3}\right)\right\} + \frac{3\alpha^2}{(1-n)(6+6n+4m+2mn)}\left\{2n^2-2n+2(n+2)^2\frac{m}{3}-\frac{1}{(1+\omega)}\left(-1+4n+n^2+\frac{2m}{3}+\frac{7mn}{3}+mn^2-\frac{8m^2}{9}-\frac{8m^2n}{9}-\frac{2m^2n^2}{9}\right)\right\}\right] \quad (34)$$

where ω is the EoS parameter and $p = \omega\rho$.

4. Conclusion

We consider in this paper a Bianchi Type-III universe in the metric *f(R)* gravity. The non vacuum solutions corresponding to the isotropic perfect fluid are found under some assumptions. The first assumption is that the expansion scalar θ is proportional to the shear scalar. It gives $C = A^n$, where A, C are the metric co-efficients and n is an arbitrary constant. Secondly, the power law relation between F and a is used to find the solution.

In this model we have found that the scale factors are vanish at $T = 0$ which shows that the space time exhibits point type singularity and continues to expand till $T \to \infty$. The physical parameters H, θ, σ are all infinite at early times and approaches to zero at later times. The energy density and pressure of the fluid are related to the EoS parameter and it characterize the dark energy into different expansion histories. It is observe that our solution favors only the phantom region because energy density is negative for other region which is not feasible. This phantom region of dark energy also supported by recent Supernovae data. For this model, it is also observe that the universe does not achieve isotropy and the universe expands continuously with non-zero shear scalar.

Received December 31, 2016; Accepted January 15, 2017

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