Article

An Inequality for the Fejér-Lanczos Factors

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Abstract

We find, via numerical experimentation, an inequality satisfied by the Fejér-Lanczos factors. Let's remember that those factors are important for a correct differentiation of Fourier series.

Keywords: Differentiation of Fourier series, Fejér-Lanczos coefficients.

1. Introduction

The Fourier series can be differentiated via the symmetric derivative [1] or applying the orthogonal derivative [2], in both cases the result is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sigma_{n,k} \frac{d}{dx}[a_k \cos(kx) + b_k \sin(kx)], \qquad (1)$$

with the important participation of the Fejér [3]-Lanczos [1, 4, 5] factors [6]:

$$\sigma_{n,k} \equiv \frac{\sin(\frac{k}{n}\pi)}{\frac{k}{n}\pi}, \qquad k = 0, 1, \dots, n. \qquad (2)$$

By numerical experimentation we calculate (2) for $n = 2, 3, \dots, 21$ and from the corresponding Table of values we obtained the inequality:

$$\sigma_{n+1,k+1} < \sigma_{n,k} < \sigma_{n+1,k}, \qquad k = 1, \dots, n, \qquad (3)$$

which is not evident from the definition (2). In Sec. 2 we consider the trigonometric inequality associated to (3).

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2. Fejér-Lanczos coefficients

The relation (3) means:

$$\frac{\sin(\frac{k+1}{n+1}\pi)}{\frac{k+1}{n+1}\pi} < \frac{\sin(\frac{k}{n}\pi)}{\frac{k}{n}\pi} < \frac{\sin(\frac{k}{n+1}\pi)}{\frac{k}{n+1}\pi}, \tag{4}$$

which is equivalent to the conditions:

$$n\sin\left(\frac{k}{n}\pi\right) < (n+1)\sin\left(\frac{k}{n+1}\pi\right), \tag{5}$$

$$(n+1)\,k\sin\left(\frac{k+1}{n+1}\pi\right) < n\,(k+1)\sin\left(\frac{k}{n}\pi\right). \tag{6}$$

In (5) we can employ the Taylor series, then the validity of this inequality is consequence from the property:

$$\left[\frac{1}{n^{4m+2}} - \frac{1}{n^{4m+2}}\right] > \frac{(k\pi)^2}{(4m+4)(4m+5)}\left[\frac{1}{n^{4m+4}} - \frac{1}{(n+1)^{4m+4}}\right], \quad m = 0, 1, 2, \dots \ , \quad k = 0, \dots, n\,; \tag{7}$$

similarly, (6) is correct because it is possible to verify the constraint:

$$\left[\left(\frac{k+1}{n+1}\right)^{4m+2} - \left(\frac{k}{n}\right)^{4m+2}\right] > \frac{\pi^2}{(4m+4)(4m+5)}\left[\left(\frac{k+1}{n+1}\right)^{4m+4} - \left(\frac{k}{n}\right)^{4m+4}\right],$$

$$k = 0, \dots, n\ , \quad m = 0, 1, 2, \dots. \tag{8}$$

It is interesting to note that (8) gives a good upper bound for π, in fact, into (8) we use $m = 0$, and after simplification we select $k = n$ to obtain that $\pi < \sqrt{10} = 3.1622\,7766$, obtained by the Indian mathematician Brahmagupta [7].

Thus we see that a numerical study of the $\sigma_{n,k}$ factors implies the non-trivial trigonometric inequality (4). The Fejér-Lanczos multipliers are important in the differentiation of Fourier series, to accelerate the convergence of certain series, and to reduce the Gibbs phenomenon [6].

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References

1. C. Lanczos, *Applied analysis*, Dover, New York (1988) Chap. 4.
2. R. Cruz-Santiago, J. López-Bonilla, R. López-Vázquez, *Differentiation of Fourier series via orthogonal derivative*, J. Inst. Sci. Tech. (Nepal) **20**, No. 2 (2015) 113-114.

3. L. Fejér, *Untersuchungen über Fouriersche reihen,* Math. Annalen **58** (1904) 51-69.
4. C. Lanczos, *Discourse on Fourier series,* Oliver & Boyd, Edinburgh (1966) Chap. 1.
5. C. Lanczos, *Linear differential operators,* SIAM, Philadelphia, USA (1996) Chap. 2.
6. A. J. Jerri, *Lanczos-like σ-factors for reducing the Gibbs phenomenon in general orthogonal expansions and other representations,* J. Comp. Anal. Appls. **2**, No. 2 (2000) 1-17.
7. P. Beckmann, *A history of Pi,* St. Martin's Press, New York (1975) Chap. 2.