Article

On Exponential Operators

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Abstract

We study $e^{\xi(A+B)}$ under several conditions between A and B with their commutator, which leads to the factorizations of Fujiwara, Beauregard and García-Arévalo.

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1. Introduction

It is very known the relation [1]:

$$e^{\xi(A+B)} = e^{\xi A}\, e^{\xi B}\, e^{-\frac{\xi^2}{2}[A,B]}, \tag{1}$$

where ξ is an arbitrary parameter, and the operators A and B commute with their commutator:

$$[A,[A,B]] = 0\,, \qquad [B,[A,B]] = 0. \tag{2}$$

Then it is natural to search expressions for $e^{\xi(A+B)}$ under several conditions, for example:

$$[A,[A,B]] = 2\gamma\, A, \qquad [B,[A,B]] = -2\gamma\, B, \tag{3}$$

or

$$[A,[A,B]] = 0, \qquad [B,[A,B]] = k. \tag{4}$$

The case (3) is analyzed in [2-5], and the option (4) is considered in [2, 6]; in Sec. 2 we study the results of these papers, and Sec. 3 contains applications of these results.

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2. Exponential operators

From (1) with the conditions (2):

$$e^{\xi A}\, e^{\xi B}\, e^{-\frac{\xi^2}{2}[A,B]} = e^{\xi(A+B)} = e^{\xi B}\, e^{\xi A}\, e^{\frac{\xi^2}{2}[A,B]},$$

therefore:

$$e^{\xi A}\, e^{\xi B} = e^{\xi B}\, e^{\xi A}\, e^{\xi^2[A,B]}\,. \tag{5}$$

Besides:

$$e^{\xi(A+B)} = e^{\xi A}\, e^{\frac{\xi}{2}B}\, e^{\frac{\xi}{2}B}\, e^{-\frac{\xi^2}{2}[A,B]} \overset{(5)}{=} e^{\frac{\xi}{2}B}\, e^{\xi A}\, e^{\frac{\xi^2}{2}[A,B]} e^{-\frac{\xi^2}{2}[A,B]}\, e^{\frac{\xi}{2}B},$$

hence:

$$e^{\xi(A+B)} = e^{\frac{\xi}{2}B}\, e^{\xi A}\, e^{\frac{\xi}{2}B}, \tag{6}$$

under the constraints (2).

On the other hand, García-Arévalo [4] employ (3) and parameter differentiation [7, 8] to deduce the factorization:

$$e^{\xi(A+B)} = \exp\left[\frac{1}{\sqrt{\gamma}}\tan\left(\frac{\xi}{2}\sqrt{\gamma}\right)B\right]\exp\left[\frac{1}{\sqrt{\gamma}}\sin(\xi\sqrt{\gamma})\,A\right]\exp\left[\frac{1}{\sqrt{\gamma}}\tan\left(\frac{\xi}{2}\sqrt{\gamma}\right)B\right]. \tag{7}$$

For the case $\gamma \longrightarrow 0$ the expressions (3) and (7) reproduce (2) and (6). If $\xi = 2\,\lambda$ and $\gamma = 1$, then (7) gives the Fujiwara [2]-Beauregard [3] formula [5]:

$$e^{2\lambda(A+B)} = e^{B\,\tan\lambda}\; e^{A\,\sin(2\lambda)}\, e^{B\,\tan\lambda}\,. \tag{8}$$

For the conditions (4) we have the García-Arévalo [6] relation:

$$e^{\xi(A+B)} = e^{\frac{\xi^3}{3}k}\, e^{\xi A}\, e^{\xi B}\, e^{-\frac{\xi^2}{2}[A,B]}, \tag{9}$$

and we recover (1) and (2) when $k = 0$; the factorization (9) can be written in the form:

$$e^{\xi(A+B)} = e^{-\frac{\xi^3}{6}k}\, e^{-\frac{\xi^2}{2}[A,B]}\, e^{\xi A}\, e^{\xi B}\,. \tag{10}$$

With the procedure indicated in [5] is possible to obtain the following expression:

$$e^{\xi(A+B)} = e^{-\frac{\xi^3}{24}k}\, e^{\frac{\xi}{2}B}\, e^{\xi A}\, e^{\frac{\xi}{2}B}, \tag{11}$$

as an alternative to (9); if $k = 0$ then (11) implies (6).

3. Special cases

Now we shall realize some applications of the results exhibited at Sec. 2:

a). $\xi = u, \quad A = \omega \frac{d^2}{dx^2}, \quad B = \frac{x^2}{\omega},$ then:

$$[A, B] = 2\left(1 + 2\,x\frac{d}{dx}\right), \qquad \big[A, [A, B]\big] = 8\,A, \qquad \big[B, [A, B]\big] = -8\,B,$$

in according with (3) for $\gamma = 4$, hence (7) implies the following expression of Fujiwara [2]-Beauregard [3]:

$$e^{u\,(\omega \frac{d^2}{dx^2} + \frac{x^2}{\omega})} = e^{\frac{x^2}{2\omega}\tan u}\, e^{\frac{\omega}{2}\sin(2u)\frac{d^2}{dx^2}}\, e^{\frac{x^2}{2\omega}\tan u}, \tag{12}$$

which is useful to determine the time evolution operator for a one-dimensional harmonic oscillator.

b). $A = a\,x^2, \; B = \frac{d}{dx},$ thus:

$$[A, B] = -2\,a\,x, \qquad \big[A, [A, B]\big] = 0, \qquad \big[B, [A, B]\big] = -2\,a = k,$$

verifying (4), then from (10):

$$e^{\xi\,(a\,x^2 + \frac{d}{dx})} = e^{a\,\xi\left(x^2 + \xi\,x + \frac{\xi^2}{3}\right)}\, e^{\xi\frac{d}{dx}}. \tag{13}$$

c). $\xi = u, \quad A = \frac{d^2}{dx^2}, \quad B = 2\omega\,x,$ therefore:

$$[A, B] = 4\,\omega\,\frac{d}{dx}, \qquad \big[A, [A, B]\big] = 0, \qquad \big[b, [a, b]\big] = -8\,\omega^2 = k,$$

in according with (4), hence (11) implies the relation of Fujiwara [2]:

$$e^{u\,(\frac{d^2}{dx^2} + 2\,\omega\,x)} = e^{\frac{u^3}{3}\omega^2}\, e^{u\,\omega\,x}\, e^{u\frac{d^2}{dx^2}}\, e^{u\,\omega\,x}. \tag{14}$$

We consider that is important to study the splitting of $e^{\xi(A+B)}$ for several conditions between A and B with their commutator.

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