Article

Spacetime with a Constant Spinor Field

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Abstract

Taub used spinor analysis to prove that a spacetime admitting a covariantly constant spinor field is Petrov type N. Here we employ the Newman-Penrose formalism to give a simple proof of this interesting Taub's theorem.

Keywords: Constant spin-vector, Taub's theorem, Newman-Penrose formalism.

1. Introduction

We accept that o_A is a constant spinor field:

$$\nabla_\mu \, o_A = 0, \tag{1}$$

and we construct the null tetrad [1]:

$$l^\mu \;\leftrightarrow\; o^A o^{\dot{B}}, \quad n^\mu \;\leftrightarrow\; \iota^A \iota^{\dot{B}}, \quad m^\mu \;\leftrightarrow\; o^A \iota^{\dot{B}}, \quad \overline{m}^\mu \;\leftrightarrow\; \iota^A o^{\dot{B}}, \quad o_A \iota^A = 1, \tag{2}$$

then (1) gives the constraints [2]:

$$\hat{L}\, o_A = 0, \qquad \hat{L} = D,\ \Delta,\ \delta,\ \bar{\delta}. \tag{3}$$

In Sec. 2 we show that (3) and the Newman-Penrose (NP) equations [3] imply that l_μ is a 4-degenerate principal direction of the Weyl tensor, hence the spacetime is type N [4-6], in accordance with Taub [7].

2. Constant spinor field

We have the relations [1, 2]:

$$D o_A = \varepsilon\, o_A - \kappa\, \iota_A, \quad \Delta o_A = \gamma\, o_A - \tau\, \iota_A, \quad \delta o_A = \beta\, o_A - \sigma\, \iota_A, \quad \bar{\delta} o_A = \alpha\, o_A - \rho\, \iota_A, \tag{4}$$

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then from (3):

$$\kappa = \sigma = \varepsilon = \gamma = \rho = \tau = \alpha = \beta = 0, \tag{5}$$

that we can employ into the NP equations [1, 3] to obtain:

$$\psi_a = 0, \qquad a = 0, \ldots, 3, \tag{6}$$

which means a spacetime with Petrov type N [4-6]. Besides, $R = 0$ and:

$$\phi_{ab} = 0 \qquad \text{except possibly} \quad \phi_{22}\,, \tag{7}$$

therefore:

$$R_{\mu\nu} = -2\,\phi_{22}\,l_\mu\,l_\nu\,, \tag{8}$$

thus the Ricci tensor is type $[4N]_{[2]} = [(112)]$ in the Churchill-Plebañski classification [8-10].

The Taub's condition (1) implies that l_μ is a constant null vector field:

$$\nabla_\nu\,l_\mu = 0, \tag{9}$$

which can be applied in the known property [11]:

$$(\nabla_\lambda \nabla_\nu - \nabla_\nu \nabla_\lambda)\,l_\mu = R_{\theta\mu\lambda\nu}\,l^\theta\,, \tag{10}$$

to deduce that l_μ is a 4-degenerate principal direction of the conformal tensor [6]:

$$C_{\theta\mu\lambda\nu}\,l^\theta = 0\,, \tag{11}$$

where we use (8) and $R = 0$.

It is possible to give a spinor proof of this Taub's theorem, in fact, we have the expressions [12]:

$$\square_{AB}\,o_C = \psi_{ABCD}\,o^D + \frac{1}{24} R\,(\varepsilon_{AC}\,o_B + \varepsilon_{BC}\,o_A), \qquad \square_{AB}\,o^B = \frac{1}{8} R\,o_A\,, \tag{12}$$

where we can employ (1) in the form $\nabla_{C\dot{D}}\,o_A = 0$ to obtain $R = 0$ and that o_A is a principal spinor of the Weyl spinor [13]:

$$\psi_{ABCD}\,o^D = 0, \tag{13}$$

equivalent to (6) and (11). Besides [11]:

$$\Box_{AB}\, o_{\dot{C}} = \phi_{AB\dot{C}\dot{D}}\, o^{\dot{D}} \overset{(1)}{=} 0\,, \tag{14}$$

which implies (7) and (8).

We see that $\nabla_\mu\, o_A = 0$ leads to $\nabla_\mu\, l_\nu = 0$, however, the inverse situation is that a constant null vector can have $\nabla_\mu\, o_A \neq 0$. Here the Taub's result was proved with NP and spinor tools.

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