Spacetime with a Constant Spinor Field

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Abstract
Taub used spinor analysis to prove that a spacetime admitting a covariantly constant spinor field is Petrov type N. Here we employ the Newman-Penrose formalism to give a simple proof of this interesting Taub’s theorem.

Keywords: Constant spin-vector, Taub’s theorem, Newman-Penrose formalism.

1. Introduction

We accept that \(o_A\) is a constant spinor field:

\[
\nabla_\mu o_A = 0,
\]

and we construct the null tetrad [1]:

\[
\begin{align*}
  l^\mu &\leftrightarrow o^A o^B, \\
n^\mu &\leftrightarrow \iota^A \iota^B, \\
m^\mu &\leftrightarrow o^A \iota^B, \\
\bar{m}^\mu &\leftrightarrow \iota^A o^B, \\
o_A \iota^A &= 1,
\end{align*}
\]

then (1) gives the constraints [2]:

\[
\hat{L} o_A = 0, \quad \hat{L} = D, \Delta, \delta, \bar{\delta}.
\]

In Sec. 2 we show that (3) and the Newman-Penrose (NP) equations [3] imply that \(l_\mu\) is a 4-degenerate principal direction of the Weyl tensor, hence the spacetime is type N [4-6], in accordance with Taub [7].

2. Constant spinor field

We have the relations [1, 2]:

\[
\begin{align*}
  Do_A &= \varepsilon o_A - \kappa \iota_A, \\
  \Delta o_A &= \gamma o_A - \tau \iota_A, \\
  \delta o_A &= \beta o_A - \sigma \iota_A, \\
  \bar{\delta} o_A &= \alpha o_A - \rho \iota_A,
\end{align*}
\]

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then from (3):
\[ \kappa = \sigma = \varepsilon = \gamma = \rho = \tau = \alpha = \beta = 0, \] (5)
that we can employ into the NP equations [1, 3] to obtain:
\[ \psi_a = 0, \quad a = 0, \ldots, 3, \] (6)
which means a spacetime with Petrov type N [4-6]. Besides, \( R = 0 \) and:
\[ \phi_{ab} = 0 \quad \text{except possibly} \quad \phi_{22}, \] (7)
therefore:
\[ R_{\mu \nu} = -2 \phi_{22} \ l_{\mu} \ l_{\nu}, \] (8)
thus the Ricci tensor is type \([4N]_2 = [(112)] \) in the Churchill-Plebański classification [8-10].

The Taub’s condition (1) implies that \( l_{\mu} \) is a constant null vector field:
\[ \nabla_{\nu} \ l_{\mu} = 0, \] (9)
which can be applied in the known property [11]:
\[ (\nabla_{\lambda} \nabla_{\nu} - \nabla_{\nu} \nabla_{\lambda}) \ l_{\mu} = R_{\theta \mu \lambda \nu} \ l^{\theta}, \] (10)
to deduce that \( l_{\mu} \) is a 4-degenerate principal direction of the conformal tensor [6]:
\[ C_{\theta \mu \lambda \nu} \ l^{\theta} = 0, \] (11)
where we use (8) and \( R = 0 \).

It is possible to give a spinor proof of this Taub’s theorem, in fact, we have the expressions [12]:
\[ \Box_{AB} \ o_C = \psi_{ABCD} \ o^D + \frac{1}{24} R (\varepsilon_{AC} \ o_B + \varepsilon_{BC} \ o_A), \quad \Box_{AB} \ o^B = \frac{1}{8} R \ o_A, \] (12)
where we can employ (1) in the form \( \nabla_{CD} \ o_A = 0 \) to obtain \( R = 0 \) and that \( o_A \) is a principal spinor of the Weyl spinor [13]:
\[ \psi_{ABCD} \ o^D = 0, \] (13)
equivalent to (6) and (11). Besides [11]:
\[ \Box_{AB} \phi^C_{ABCD} \phi^D = 0 , \]  

which implies (7) and (8).

We see that \( \nabla \mu o_A = 0 \) leads to \( \nabla \mu l_v = 0 \), however, the inverse situation is that a constant null vector can have \( \nabla \mu o_A \neq 0 \). Here the Taub’s result was proved with NP and spinor tools.

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