Article

Killing Magnetic Curves in Three Dimensional Isotropic Space

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Abstract

In this paper, we study and classify the magnetic curves in the isotropic 3-space associated to a Killing vector field $V = v_i\partial_i$ with $\partial_i = \frac{\partial}{\partial x_i}$ and $v_i \in \mathbb{R}$, $i = 1, 2, 3$.

Keywords: Magnetic trajectory, Lorentz force, killing vector field, isotropic space.

1 Introduction

Let (N, g) denote a Riemannian manifold and F a closed 2-form. F is said to be a *magnetic field.* The *Lorentz force* of a magnetic background (M, g, F) is the skew symmetric $(1,1)$ −type tensor field ϕ on N satisfying

$$g(\phi(X), Y) = F(X, Y) \tag{1.1}$$

for any X, Y tangent to N. Hence a *magnetic curve* associated to F is a smooth curve c on N satisfying

$$\nabla_{c'}c' = \Phi(c'), \tag{1.2}$$

where ∇ is the Levi-Civita connection of g. The equation (1.2) is known as the *Lorentz equation.*

Since F is skew symmetric in (1.1), the magnetic curve c has constant speed, i.e. $g(c', c') = \lambda = const$. In the particular case $\lambda = 1$, it is called a *normal magnetic curve.* We consider only the normal magnetic curves all over this paper.

The first study of magnetic fields was treated on Riemannian surfaces (see e.g. (7, 25)), then in 3-dimensional context, on $\mathbb{E}^3$ (15), $\mathbb{E}^3_1$ (16), $\mathbb{S}^3$ (8), $\mathbb{S}^2 \times \mathbb{R}$ (23) etc. For more study of the magnetic curves on the (semi-) Riemannian manifolds, we refer to (1, 2, 7, 10, 12, 13, 17, 21, 22).

On the other hand, the isotropic 3-space $\mathbb{I}^3$ is a Cayley–Klein space defined from a 3-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure which is an ordered triple (w, f_1, f_2), where w is a plane in $P(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in w, see (18)-(20).

The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the absolute plane w is given by $X_0 = 0$ and the absolute lines f_1, f_2 by $X_0 = X_1 + iX_2 = 0$, $X_0 = X_1 - iX_2 = 0$. The intersection point $F(0 : 0 : 0 : 1)$ of these two lines is called the absolute point. The affine coordinates are obtained by $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, $x_3 = \frac{X_3}{X_0}$.

In this paper, our aim is to classify the magnetic curves in $\mathbb{I}^3$. In this manner we derive some classifications for the magnetic curves and N−magnetic curves with constant curvature (see Definition 4.1) associated to the Killing vector field $V = v_1\partial_1 + v_2\partial_2 + v_3\partial_3$ in $\mathbb{I}^3$, where $\partial_i = \frac{\partial}{\partial x_i}$ are orthonormal basis vector fields, $i = 1, 2, 3$.

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2 Preliminaries

In this section we provide the fundamental notions on isotropic spaces from (3, 4), (26)- (31).

The *isotropic distance* in $\mathbb{I}^3$ of two points $x = (x_i)$ and $y = (y_i)$, $i = 1, 2, 3$, is defined as

$$\|x - y\|_{\mathbb{I}} = \sqrt{\sum_{j=1}^{2} (y_j - x_j)^2}. \tag{2.1}$$

The lines in x_3-direction are called *isotropic lines*. The plane containing an isotropic line is said to be an *isotropic plane*. Other planes are *non-isotropic*.

The *isotropic scalar product* between two vectors $a = (a_i)$ and $b = (b_i)$ in $\mathbb{I}^3$ is given by

$$\langle a, b\rangle_{\mathbb{I}} = \begin{cases} a_1 b_1 + a_2 b_2, & a_i \neq 0 \text{ or } b_i \neq 0, \ (i = 1, 2), \\ a_3 b_3, & a_i = b_i = 0, \ (i = 1, 2). \end{cases} \tag{2.2}$$

The *cross product* of two vectors $a = (a_i)$ and $b = (b_i)$ in $\mathbb{I}^3$ can be defined by

$$a \times_{\mathbb{I}} b = \begin{vmatrix} e_1 & e_2 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{2.3}$$

for $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$. It is easy to check that

$$\langle a \times_{\mathbb{I}} b, c\rangle_{\mathbb{I}} = \det(a, b, \widetilde{c}),$$

where $\widetilde{c}$ means the projection of c on the Euclidean (x_1, x_2) −plane. For more details, see (5).

Let $c : I \to \mathbb{I}^3$, $I \subset \mathbb{R}$, be a curve parameterized by the arc length. It is called *admissible* if it has no tangent vector field in x_3-direction. An admissible curve can be given in the form

$$c(s) = (x(s), y(s), z(s)), \ \widetilde{c}(s) = (x(s), y(s)),$$

where $\widetilde{c}' = (x'(s), y'(s)) \neq 0$. The curvature $\kappa(s)$ and the torsion $\tau(s)$ are respectively defined by

$$\begin{aligned} \kappa(s) &= \det(\widetilde{c}'(s), \widetilde{c}''(s)) \\ \tau(s) &= \frac{\det(c'(s), c''(s), c'''(s))}{\kappa^2(s)}, \ \kappa(s) \neq 0, \end{aligned}$$

and the associated trihedron is given by

$$\begin{cases} T = (x'(s), y'(s), z'(s)), \\ N = \frac{1}{\kappa(s)}((x''(s), y''(s), z''(s))), \\ B = (0, 0, 1). \end{cases} \tag{2.4}$$

For such vector fields the following Frenet's formulas hold

$$T' = \kappa N, \ N' = -\kappa T + \tau B, \ B' = 0. \tag{2.5}$$

3 Killing Magnetic Trajectories in $\mathbb{I}^3$

Let (N, g) be a Riemannian manifold and X a vector field on N. If $L_X g = 0$ then X is called a *Killing vector field*, where L denotes the Lie derivative with respect to X. It is easily seen that X is a Killing vector field on N if and only if

$$g\left(\nabla_Y X, Z\right) + g\left(\nabla_Z X, Y\right) = 0,$$

where ∇ is the Levi-Civita connection of g.

The 2-forms on 3-dimensional manifolds may correspond to the vector fields via the Hodge $\star$ operator and the volume form dv_g of the manifold. Hence, we can consider the Killing magnetic fields associated the Killing vector fields.

Note that the cross product of any vector fields X, Y on N is defined as

$$g\left(X \times Y, Z\right) = dv_g\left(X, Y, Z\right),$$

where X, Y, Z tangent to N.

Let $F_V = \iota_V dv_g$ be the Killing magnetic field corresponding to the Killing vector field V, where ι denotes the inner product. Then, the Lorentz force of F_V is (see e.g. (8, 15))

$$\Phi\left(X\right) = V \times X. \tag{3.1}$$

From (1.2) and (3.1), the Lorentz force of $\left(\mathbb{I}^3, \langle,\rangle_{\mathbb{I}}, F_V\right)$ is

$$c'' = V \times_{\mathbb{I}} c', \tag{3.2}$$

where V is a Killing vector field on $\mathbb{I}^3$. We call c *Killing magnetic curve*.

Now let c be a curve in $\mathbb{I}^3$, parametrized by the arc length and given in the coordinate form

$$c\left(s\right) = \left(x(s), y\left(s\right), z\left(s\right)\right),\ s \in I \subset \mathbb{R}, \tag{3.3}$$

where x, y and z are smooth functions satisfying the initial conditions:

$$x\left(0\right) = x_0,\ x'\left(0\right) = X_0,\ y\left(0\right) = y_0,\ y'\left(0\right) = Y_0 \text{ and } z\left(0\right) = z_0,\ z'\left(0\right) = Z_0. \tag{3.4}$$

Remark 3.1. c is a non-isotropic line in $\mathbb{I}^3$ when $V \equiv 0$. Afterwards we assume that V is not identically zero.

By the following result, we classify the normal magnetic trajectories associated to the Killing vector $V = v_1\partial_1 + v_2\partial_2 + v_3\partial_3$ in $\mathbb{I}^3$, where $\partial_i = \frac{\partial}{\partial x_i}$ and $v_1, v_2, v_3 \in \mathbb{R}$.

Theorem 3.1. *Let c be a normal magnetic curve associated to the Killing vector $V = v_1\partial_1 + v_2\partial_2 + v_3\partial_3$ in $\mathbb{I}^3$ with the initial conditions (3.4). Then:*

(i) *If* $V = (v_1, v_2, 0)$,

$$c\left(s\right) = \left(\frac{v_2 Z_0}{2}s^2 + X_0 s + x_0,\ -\frac{v_1 Z_0}{2}s^2 + Y_0 s + y_0,\ Z_0 s + z_0\right); \tag{3.5}$$

(ii) *if* $V = (v_1,\ v_2,\ v_3 \neq 0)$,

$$c\left(s\right) = \Bigg(\left(\lambda_1 - x_0\right)\cos\left(v_3 s\right) + \left(\lambda_2 - y_0\right)\sin\left(v_3 s\right) + \frac{Z_0 v_1}{v_3}s + \lambda_1, \tag{3.6}$$

$$\left(\lambda_2 - y_0\right)\cos\left(v_3 s\right) - \left(\lambda_1 - x_0\right)\sin\left(v_3 s\right) + \frac{Z_0 v_2}{v_3}s + \lambda_2, Z_0 s + z_0\Bigg),$$

where $\lambda_1 = x_0 + \frac{Y_0 - Z_0 \frac{v_1}{v_3}}{v_3}$ and $\lambda_2 = y_0 + \frac{X_0 - Z_0 \frac{v_2}{v_3}}{v_3}$.

Proof. If c is a normal magnetic trajectory in $\mathbb{I}^3$, then it is a solution of (3.2). We have to consider two cases for the proof.

Case 1. $V = (v_1, v_2, 0)$. Then it follows from (2.3) and (3.2) that

$$\begin{cases} x'' = v_2 z', \\ y'' = -v_1 z', \\ z'' = 0,\ z = Z_0 s + z_0. \end{cases} \tag{3.7}$$

After considering the initial conditions (3.4) into (3.7) , c derives the form (3.5) . This implies the statement (i).

Case 2. $V = (v_1,\ v_2,\ v_3 \neq 0)$. Hence, from (2.4) and (3.2) we have

$$\begin{cases} x'' = -Z_0 v_2 + v_3 y', \\ y'' = Z_0 v_1 - v_3 x'. \end{cases} \tag{3.8}$$

We may formulate the Cauchy problem associated to system (3.8) and the initial conditions (3.4) as follows:

$$\begin{cases} x''' = Z_0 v_1 v_3 - v_3^2 x', \\ y''' = Z_0 v_2 v_3 - v_3^2 y'. \end{cases} \tag{3.9}$$

Solving (3.9) , we obtain

$$\begin{aligned} x(s) &= \frac{Z_0 v_1}{v_3} s + \frac{X_0 - Z_0 \frac{v_1}{v_3}}{v_3} \sin(v_3 s) + \frac{Y_0 - Z_0 \frac{v_1}{v_3}}{v_3} \cos(v_3 s) + \\ &\quad + x_0 + \frac{Y_0 - Z_0 \frac{v_1}{v_3}}{v_3}, \\ y(s) &= \frac{Z_0 v_2}{v_3} s - \frac{Y_0 - Z_0 \frac{v_1}{v_3}}{v_3} \sin(v_3 s) + \frac{X_0 - Z_0 \frac{v_1}{v_3}}{v_3} \cos(v_3 s) + \\ &\quad + y_0 + \frac{X_0 - Z_0 \frac{v_2}{v_3}}{v_3}. \end{aligned}$$

which completes the proof.

4 $N-$Magnetic and $B-$Magnetic Curves in $\mathbb{I}^3$

Definition 4.1. (10) Let $c : I \subset \mathbb{R} \longrightarrow M$ be a curve in an oriented 3-dimensional Riemannian manifold (M, g) and F be a magnetic field on M. The curve α is an $N-$*magnetic curve* (respectively $B-$*magnetic curve*) if the normal vector field N (respectively the binormal vector field B) of the curve satisfies the Lorentz force equation, i.e., $\nabla_{\dot\gamma} N = \Phi(N) = V \times N$ (respectively $\nabla_{\dot\gamma} B = \Phi(B) = V \times B$).

In this section in order to obtain certain results, we assume that the curve c has nonzero constant curvature κ_0.

Let consider the arc length curve c in $\mathbb{I}^3$, parametrized by

$$c(s) = (x(s), y(s), z(s)), \tag{4.1}$$

$$\left\{\begin{array}{l} x(0)=x_0, x'(0)=X_0, x''(0)=T_0, \\ y(0)=y_0,\ y'(0)=Y_0,\ y''(0)=U_0, \\ z(0)=z_0,\ z'(0)=Z_0,\ z''(0)=W_0. \end{array}\right. \tag{4.2}$$

Next we classify the $N-$magnetic curves with constant curvature $\kappa(s)=\kappa_0\neq 0$, corresponding to the Killing vector $V=v_1\partial_1+v_2\partial_2+v_3\partial_3$ in $\mathbb{I}^3$, where $\partial_i=\frac{\partial}{\partial x_i}$ and $v_1,v_2,v_3\in\mathbb{R}$.

Theorem 4.1. *Let c be a normal $N-$magnetic trajectory with constant curvature κ_0 associated to the Killing vector $V=v_1\partial_1+v_2\partial_2+v_3\partial_3$ in $\mathbb{I}^3$, satisfying the initial conditions* (4.2)*. Then c is one of the following forms:*
(i) *If* $v_3=0$,

$$\begin{aligned} c(s) &= \left(\frac{W_0v_2}{6}s^3+\frac{T_0}{2}s^2+X_0s+x_0, -\frac{W_0v_1}{6}s^3+\frac{U_0}{2}s^2+Y_0s+y_0,\right. \\ &\quad \left.\frac{W_0}{2}s^2+Z_0s+z_0\right); \end{aligned}$$

(ii) *Otherwise,*

$$\begin{aligned} c(s) &= \left(\frac{v_1W_0}{2v_3}s^2+(X_0-U_0)s-T_0\cos(v_3s)+U_0\sin(v_3s)+x_0+T_0,\right. \\ &\quad \frac{v_2W_0}{2v_3}s^2+(Y_0+T_0)s-U_0\cos(v_3s)-T_0\sin(v_3s)+y_0+U_0, \\ &\quad \left.\frac{W_0}{2}s^2+Z_0s+z_0\right). \end{aligned}$$

Proof. Assume that c is a normal $N-$magnetic curve corresponding to the Killing vector $V=v_1\partial_1+v_2\partial_2+v_3\partial_3$ in $\mathbb{I}^3$. By the Definition 4.1, we have

$$\nabla_{\dot{\gamma}}N=\Phi(N)=V\times_{\mathbb{I}}N, \tag{4.3}$$

where $N=\frac{1}{\kappa_0}\left(x''(s),y''(s),z''(s)\right)$.
Case 1. $v_3=0$. By using the cross product in $\mathbb{I}^3$ and (4.3), we get

$$\left\{\begin{array}{l} x'''=v_2z'', \\ y'''=-v_1z'', \\ z'''=0,\ z=\frac{W_0}{2}s^2+Z_0s+z_0. \end{array}\right. \tag{4.4}$$

By solving the Cauchy problem associated to system (4.4) and the initial conditions (4.2), we obtain

$$\begin{aligned} x(s) &= \frac{W_0v_2}{6}s^3+\frac{T_0}{2}s^2+X_0s+x_0, \\ y(s) &= -\frac{W_0v_1}{6}s^3+\frac{U_0}{2}s^2+Y_0s+y_0, \end{aligned}$$

which implies the first statement.
Case 2. $v_3\neq 0$. From (2.3) and (4.3), we derive

$$\left\{\begin{array}{l} x'''=v_2z''-v_3y'', \\ y'''=-v_1z''+v_3x''. \end{array}\right. \tag{4.5}$$

By solving the Cauchy problem associated to system (4.5) and the initial conditions (4.2), we obtain

$$\begin{aligned} x(s) &= \frac{v_1 W_0}{2v_3} s^2 + (X_0 - U_0)s - T_0 \cos(v_3 s) + U_0 \sin(v_3 s) + x_0 + T_0 \\ y(s) &= \frac{v_2 W_0}{2v_3} s^2 + (Y_0 + T_0)s - U_0 \cos(v_3 s) - T_0 \sin(v_3 s) + y_0 + U_0, \end{aligned}$$

which completes the proof.

On the other hand, since the binormal vector field B of a curve in $\mathbb{I}^3$ is completely isotropic vector, i.e. $(0,0,1)$, the corresponding Lorentz equation becomes via Definition 4.1 as follows

$$0 = V \times_{\mathbb{I}} B.$$

This implies that either $V = 0$ or V is completely isotropic vector, i.e. $(0,0,v_3)$. Thus we obtain the following result:

Corollary 4.2. *All curves in $\mathbb{I}^3$ are $B-$magnetic curves associated to the Killing vector $V = (0,0,v_3)$.*

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