Article

On The Timelike Bertrand B-Pair Curves in 3-Dimensional Minkowski Space

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Abstract

First, we define timelike Bertrand B curves and timelike Bertrand B-pair curves according to Bishop II frame. Second, we examine the main properties between curves which are included the pair of curves. Finally, we obtain some results such that if $\varepsilon_1 > \varepsilon_2$ for $\varepsilon_1, \varepsilon_2$ the curvatures of the curve α , timelike Bertrand B curves are a spatial and if $\varepsilon_2 > \varepsilon_1$, this curves are a planar.

Keywords: Timelike Bertrand B curve, timelike Bertrand B-pair curve, Bishop II frame, Minkowski space.

1. Introduction

Unlike Euclidean geometry, Lorentz geometry includes the concept of time t is fourth dimension and so one call it not ‘space’ but ‘space time’ in general. The important step for our understanding of spacetime concerns the way in which the usual three-dimensional Euclidean geometry is embedded in the Minkowskian four-dimensional spacetime. The fact that different embedings hold for obsevers in relative uniform motion is implied by the notion of Lorentz frame. Therefore, this geometry has been remarkable in terms of the theory of relativity of mathematical physics for geometers, for example cosmology (redshift, expanding universe and big bang) and the gravitation of a single star (perihelion procession, bending of light and black holes)[1].

In the theory of curves, the second derivative of the curve may be zero that is, the curvature may vanish at some points on the curve. In this situation, an alternative frame is needed for non continously differentiable curves. Thats why, the Bishop frame was constructed by L. R. Bishop in 1975 [2]. Bishop frame is well defined and constructed in Euclidean space. In applied sciences, Bishop frame is used in engineering. This special frame has been particulary used in the study of DNA, and tubular surfaces and made in robot.

A new version of Bishop frame was first introduced and studied in Euclidean space by Yilmaz in [3]. By new version of Bishop frame, we mean that the tangent vector ξ_1 and principal normal vector ξ_2 are considered as parallel transport plane while the binormal vector B remains fixed. Afterwards, Özyilmaz investigated some characterizations of curves according to this new frame

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in Euclidean space and gave some results [4]. Also, this frame has been investigated some non Euclidean spaces such as Minkowski space analogous to Euclidean space. Ünlütürk and Yılmaz obtained the new version of Bishop frame for spacelike curves in [5]. There is also a literature containing studies of curves according to Bishop frame and its new versions (see [5-8]). Lately, Yılmaz obtained the new version of Bishop frame for timelike curves in Minkowski 3-space [9]. In Euclidean space, Bertrand B-pair curves were first introduced by Yerlikaya et al [10]. They define the Bertrand B-curve and Bertrand B-pair curves in three-dimensional Euclidean space and investigate properties of these curves and obtain some characterizations of Bertrand B-pair curves.

In this work, we construct timelike Bertrand B-curve and timelike Bertrand B-pair by using the new version of Bishop frame for a timelike curve. Then, we examined the properties of these type of curves. Finally, we obtain the characterizations of timelike Bertrand B-curve and timelike Bertrand B-pair, and give some results.

2. Preliminaries

In this study, the 3-dimensional Minkowski space $\mathbb{R}_1^3$ is the pair $\left(\mathbb{R}^3, \langle , \rangle\right)$. $\mathbb{R}^3$ is a three-dimensional real vector space equipped with a Lorentz metric (inner product),

$$\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y) \to \langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.A vector $x \neq 0$ in $\mathbb{R}_1^3$ is called spacelike, timelike or a null (lightlike), if respectively holds $\langle x, x \rangle > 0$,$\langle x, x \rangle < 0$ or $\langle x, x \rangle = 0$. Especially, the vector $x = 0$ is spacelike. If $x = (x_1, x_2, x_3) \in \mathbb{R}_1^3$ and its norm defined by

$$\|x\| = \left|\langle x, x \rangle\right|^{\frac{1}{2}} = \sqrt{\left|-x_1^2 + x_2^2 + x_3^2\right|}$$

Any given two vectors $x = (x_1, x_2, x_3)$ and $x = (x_1, x_2, x_3)$in $\mathbb{R}_1^3$ are said to be orthogonal if $\langle x, y \rangle = 0$. A vector $x = (x_1, x_2, x_3)$ in $\mathbb{R}_1^3$ which satisfies $\langle x, x \rangle = \pm 1$ is called a unit vector. Any basis $\{ f_1, f_2, f_3 \}$ on $\mathbb{R}_1^3$ is known as an orthogonal basis if the vectors $i = 1, 2, 3$ are mutually orthogonal vectors such that $\langle f_1, f_1 \rangle < 0$ and, $i = 2, 3$. We also define the vector product of *x and y* (in that order) as

$$x \times y = \begin{vmatrix} -e_1 & e_3 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}_1^3$,[11].

An arbitrary curve $\alpha=\alpha(s)$ can locally be a *spacelike*, *timelike* or *null* (*lightlike*) if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike*, *timelike or null* [12]. A non-null curve $\alpha=\alpha(s)$ is said to be parameterized by pseudo-arc length parameter *s*, if $\langle\alpha'(s),\alpha'(s)\rangle=\pm1$. In this case, the curve $\alpha=\alpha(s)$ is said to be of unit speed.

Yılmaz et al. [13] introduced a new type of Bishop frame by using binormal vector of a timelike curve as the common vector field. The Type-2 Bishop Frame is expressed as

$$\begin{bmatrix}\zeta_1'\\ \zeta_2'\\ B'\end{bmatrix}=\begin{bmatrix}0 & 0 & \in_1\\ 0 & 0 & \in_2\\ \in_1 & \in_2 & 0\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\\ B\end{bmatrix}, \tag{1}$$

the set $\{\zeta_1,\zeta_2,B\}$ is called Type-2 Bishop trihedra for a timelike curve and $\in_1,\in_2$ are called type-2 bishop curvatures. If we denote the moving Frenet frame along the curve by $\{T,N,B\}$ where T, N and B are the tangent, the principal normal and the binormal vector of the curve , respectively. Then we can express the relation between The Type-2 Bishop frame and the Frenet frame for a timelike curve by

$$\begin{bmatrix}T\\ N\\ B\end{bmatrix}=\begin{bmatrix}\sinh\theta(s) & -\cosh\theta(s) & 0\\ \cosh\theta(s) & -\sinh\theta(s) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\\ B\end{bmatrix} \tag{2}$$

where θ is the angle between the vectors N and ξ_1, $\theta(s)=\int_0^s\kappa(\sigma)d\sigma$, $\tau=\sqrt{\left|\in_2^2-\in_1^2\right|}$, $\in_1=\tau(s)\cosh\theta(s)$ and $\in_2=\tau(s)\sinh\theta(s)$.

3. Results

Definition 3.1: Let α,α^* be timelike curves in 3-dimensional Minkowski space. We denote BishopII frame along the curves α and α^* at the points $\alpha(s)$ and $\alpha^*(s^*)$ by $\{\xi_1,\xi_2,B\}$ and $\{\xi_1^*,\xi_2^*,B\}$, respectively. If B and B^* are linearly depandant for every $s\in I$, then α curve called a timelike Bertrand B-curve and α^* curve called a timelike Bertrand B-mate curve. Nevertheless, (α,α^*) called a Bertrand B-pair curve.

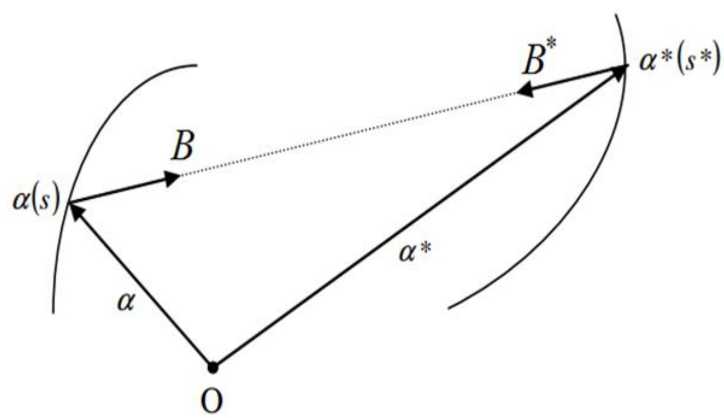

Figure 1. Bertrand B-Pair Curve.

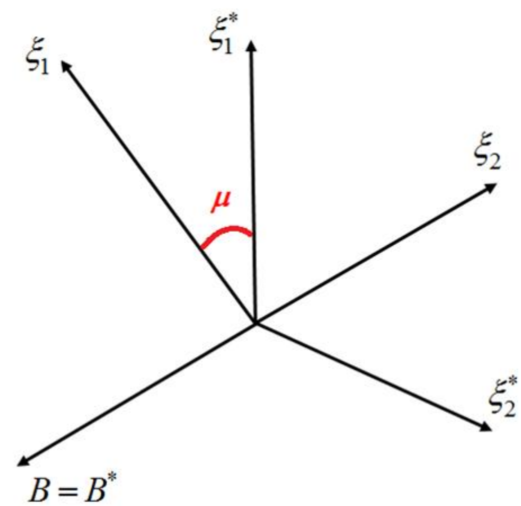

Figure 2. The Relationship Between Timelike Bertrand B-Pair Curves

Theorem 3.2: The distance between corresponding points of the timelike Bertrand B-pair in 3-dimensional Minkowski space is a constant.

Proof: From the figure1, we can write

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)B(s). \tag{3}$$

By taking the derivative of equation (3) with respect to s and using equation (1), we obtain

$$T^*(s^*)\frac{ds^*}{ds} = T(s) + \lambda'(s)B + \lambda(s)\{\varepsilon_1\xi_1 + \varepsilon_2\xi_2\}. \tag{4}$$

Using a relationship between Frenet and Bishop frame, we get

$$\left\{\sinh\theta^*(s^*)\xi_1^* - \cosh\theta^*(s^*)\xi_2^*\right\}\frac{ds^*}{ds} = \left\{\sinh\theta(s) + \lambda(s)\varepsilon_1\right\}\xi_1 + \lambda'(s)B(s) + \left\{-\cosh\theta(s) + \lambda(s)\varepsilon_2\right\}\xi_2$$

Since B and B^* are linearly dependent, we have $\left\langle \xi_1^*, B\right\rangle = \left\langle \xi_2^*, B\right\rangle = 0$. Then, we have

$$\lambda' = 0.$$

This means that λ is a nonzero constant. Subsequently, from the distance function between two points, we obtain

$$\begin{aligned} d\left(\alpha^*\left(s^*\right), \alpha(s)\right) &= \left\|\alpha(s) - \alpha^*\left(s^*\right)\right\| \\ &= \sqrt{\left|\left\langle \alpha(s) - \alpha^*\left(s^*\right), \alpha(s) - \alpha^*\left(s^*\right)\right\rangle\right|} \\ &= \sqrt{\left|\lambda^2\left\langle B, B\right\rangle\right|} \\ &= |\lambda| \ . \end{aligned}$$

In a word, $d\left(\alpha^*\left(s^*\right), \alpha(s)\right) =$ constant.

Theorem 3.3: The angle between corresponding points of both ξ_1, ξ_1^* and ξ_2, ξ_2^* vectors of a timelike Bertarand B-pairs is a constant.

Proof: Let's prove to the vector ξ_1. In terms of angle definition, we know that

$$\left\langle \xi_1, \xi_1^*\right\rangle = \left\|\xi_1\right\| . \left\|\xi_1^*\right\| \cosh\mu .$$

By both side derivative with respect to s, we have

$$\left\langle \frac{d\xi_1}{ds}, \xi_1^*\right\rangle + \left\langle \xi_1, \frac{d\xi_1^*}{ds}\right\rangle = \frac{d}{ds}\cosh\mu .$$

Additionally, using derivative equations, we get

$$\varepsilon_1\left\langle B, \xi_1^*\right\rangle + \varepsilon_1^*\frac{ds^*}{ds}\left\langle \xi_1, B^*\right\rangle = \sinh\mu\frac{d\mu}{ds} \ .$$

Since B and B^* are linearly dependent, we have both $\left\langle B, \xi_1^*\right\rangle = 0$ and $\left\langle \xi_1, B^*\right\rangle = 0$. As a result, we obtain

$$\frac{d\mu}{ds} = 0 .$$

This means that the angle between corresponding points of ξ_1 and ξ_1^* is a constant. Analogously, the angle between corresponding points of ξ_2 and ξ_2^* is also a constant.

Theorem 3.4: Let $\left\{\alpha, \alpha^*\right\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. Then, there exist the relationship between the curvatures of the curves α and α^* such that

$$\tanh\left(\operatorname{ar}\tanh\left(\frac{\varepsilon_2^*}{\varepsilon_1^*}\right)-\mu\right)=\frac{\varepsilon_2+\lambda\varepsilon_1\sqrt{\left|\varepsilon_2^2-\varepsilon_1^2\right|}}{\varepsilon_1-\lambda\varepsilon_2\sqrt{\left|\varepsilon_2^2-\varepsilon_1^2\right|}}.$$

Proof: Considering in equation (4) as constant of λ and using a relationship between frames, we get

$$\left\{\sinh\theta^*(s^*)\xi_1^*-\cosh\theta^*(s^*)\xi_2^*\right\}\frac{ds^*}{ds}=\left\{\sinh\theta(s)+\lambda\varepsilon_1\right\}\xi_1+\left\{-\cosh\theta(s)+\lambda\varepsilon_2\right\}\xi_2. \qquad (5)$$

Also, from figure2, we know that

$$\begin{aligned}\xi_1^*&=\cosh\mu\xi_1+\sinh\mu\xi_2\\ \xi_2^*&=\sinh\mu\xi_1+\cosh\mu\xi_2\end{aligned}, \qquad (6)$$

where μ is the angle between ξ_1 and ξ_1^* at the corresponding points of α and α^*. By taking into considering equations (5) and (6), we have

$$\cosh(\theta^*(s^*)-\mu)=\frac{ds}{ds^*}(\cosh\theta(s)-\lambda\varepsilon_2)$$

and

$$\sinh(\theta^*(s^*)-\mu)=\frac{ds}{ds^*}(\sinh\theta(s)+\lambda\varepsilon_1). \qquad (7)$$

Proportioning the equations (7) and arranging this equations, we get

$$\tanh(\theta^*(s^*)-\mu)=\frac{\sinh\theta(s)+\lambda\varepsilon_1}{\cosh\theta(s)-\lambda\varepsilon_2}.$$

Finally, if we write to place $\theta(s),\theta^*(s^*)$ and μ statements, then the desired expression is obtained.

Theorem 3.5: Let $\{\alpha,\alpha^*\}$ be a timelike Bertrand B-pairs in 3-dimensional Minkowski space. If respectively $\varepsilon_1,\varepsilon_2$ and $\varepsilon_1^*,\varepsilon_2^*$ the curvatures of the curves α and α^*, then the distance function between these curves is

$$\lambda=\frac{\left(\varepsilon_1\varepsilon_2^*-\varepsilon_2\varepsilon_1^*\right)\left(\sqrt{\left|\varepsilon_2^{*2}-\varepsilon_1^{*2}\right|}-\sqrt{\left|\varepsilon_2^2-\varepsilon_1^2\right|}\right)}{\left(\varepsilon_1\varepsilon_1^*+\varepsilon_2\varepsilon_2^*\right)\sqrt{\left|\varepsilon_2^2-\varepsilon_1^2\right|}\sqrt{\left|\varepsilon_2^{*2}-\varepsilon_1^{*2}\right|}}.$$

Proof: From equation (7), we have

$$\cosh(\theta^*(s^*)-\mu)=\frac{ds}{ds^*}(\cosh\theta(s)-\lambda\varepsilon_2)$$

and

$$\sinh(\theta^*(s^*)-\mu)=\frac{ds}{ds^*}(\sinh\theta(s)+\lambda\varepsilon_1).$$

Also, since $\{\alpha,\alpha^*\}$ is the timelike Bertrand B-pairs, we write

$$\alpha(s)=\alpha^*(s^*)-\lambda B^*(s^*).$$

Using the process in the proof of theorem 3, we get

$$\sinh\left(\theta(s)+\mu\right)=\frac{ds^*}{ds}\left(\sinh\theta^*\left(s^*\right)-\lambda\varepsilon_1^*\right)$$

and (8)

$$\cosh\left(\theta(s)+\mu\right)=\frac{ds^*}{ds}\left(\cosh\theta^*\left(s^*\right)+\lambda\varepsilon_2^*\right)$$

Multiplying the equations (7) and (8) and using trigonometric equations, we obtain the distance λ.

Corollary 3.6: Let $\{\alpha,\alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. If $\varepsilon_1>\varepsilon_2$ for $\varepsilon_1,\varepsilon_2$ the curvatures of the curve α, then

$$\lambda^2\tau^2\tau^{*2}-\tau^2-\tau^{*2}=0$$

is satisfied, where λ is the distance function between the curves and τ,τ^* are the torsions of curves α and α^*.

Result 3.7: Let $\{\alpha,\alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. If $\varepsilon_1>\varepsilon_2$ for $\varepsilon_1,\varepsilon_2$ the curvatures of curve α, then the curve α is planar if and only if the curve α^* is planar.

Corollary 3.8: Let $\{\alpha,\alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. If $\varepsilon_2>\varepsilon_1$ for $\varepsilon_1,\varepsilon_2$ the curvatures of curve α, then

$$\tau^2+\tau^{*2}+\lambda^2\tau^2\tau^{*2}=0$$

is satisfied, where λ is the distance function between the curves and τ,τ^* are the torsions of curves α and α^*.

Result 3.9: Let $\{\alpha,\alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. If $\varepsilon_2>\varepsilon_1$ for $\varepsilon_1,\varepsilon_2$ the curvatures of the curve α, then both the curve α and the curve α^* are planar.

Theorem 3.10: Let $\{\alpha,\alpha^*\}$ be timelike Bertrand B-pair in 3-dimensional Minkowski space. Then, there exist the relationship between the curvatures of the curves α and α^* such that

i. $\varepsilon_1=-\cosh\mu\dfrac{ds}{ds^*}\varepsilon_1^*+\sinh\mu\dfrac{ds}{ds^*}\varepsilon_2^*$

ii. $\varepsilon_2=\sinh\mu\dfrac{ds}{ds^*}\varepsilon_1^*-\cosh\mu\dfrac{ds}{ds^*}\varepsilon_2^*$

iii. $\varepsilon_1^*=-\cosh\mu\dfrac{ds^*}{ds}\varepsilon_1+\sinh\mu\dfrac{ds^*}{ds}\varepsilon_2$

iv. $\varepsilon_2^* = \sinh\mu \dfrac{ds^*}{ds}\varepsilon_1 - \cosh\mu \dfrac{ds^*}{ds}\varepsilon_2$

Proof: (i) Since B and B^* are linearly dependent, we have $\langle \xi_1, B^* \rangle = 0$. By taking the derivative with respect to s, we get

$$\left\langle \varepsilon_1 B, B^* \right\rangle + \left\langle \xi_1, \varepsilon_1^* \xi_1^* \frac{ds}{ds^*} + \varepsilon_2^* \xi_2^* \frac{ds}{ds^*} \right\rangle = 0 .$$

Using equation (5), we obtain

$$\varepsilon_1 = -\cosh\mu \frac{ds}{ds^*}\varepsilon_1^* + \sinh\mu \frac{ds}{ds^*}\varepsilon_2^* .$$

Analogously, considering equations $\langle \xi_2, B^* \rangle, \langle \xi_1^*, B \rangle, \langle \xi_2^*, B \rangle$, the proof of the statement (ii), (iii), (iv) is obvious.

In the theorem 5, from the statement both (i), (ii) and (iii), (iv), we can give in the following result:

Corollary 3.11: Let $\{\alpha, \alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. Then, there exist the relationship between the arc-parameters of the curves α and α^* such that

$$s = \int \frac{\sqrt{\left|\varepsilon_2^2 - \varepsilon_1^2\right|}}{\sqrt{\left|\varepsilon_2^{*2} - \varepsilon_1^{*2}\right|}} ds^* .$$

Remark 3.12: Let $\{\alpha, \alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. Then there exist the relationship between torsions of the curves α and $\alpha^*(s^*)$ such that

$$\tau^* ds - \tau ds^* = 0 .$$

Theorem 3.13: Let $\{\alpha, \alpha^*\}$ be a timelike Bertrand B-pair in 3-dimensional Minkowski space. Then, for the curvature centers M and M^* at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ of the curves α and α^*, the ratio

$$\frac{\left\|\alpha^*(s^*)M\right\|}{\left\|\alpha(s)M\right\|} : \frac{\left\|\alpha^*(s^*)M^*\right\|}{\left\|\alpha(s)M^*\right\|}$$

is not constant.

Proof: For the curvature center M at the point $\alpha(s)$ of the curve α, we write

$$\left\|\alpha(s)M\right\| = \left|\frac{1}{\varepsilon_1}\right| .$$

Similarly, we get

$$\left\|\alpha(s)M^{*}\right\| = \sqrt{\left|\lambda^{2} - \frac{1}{\varepsilon_1^{*2}}\right|}$$

$$\left\|\alpha^{*}\left(s^{*}\right)M^{*}\right\| = \left|\frac{1}{\varepsilon_1^{*}}\right|$$

$$\left\|\alpha^{*}\left(s^{*}\right)M\right\| = \sqrt{\left|\lambda^{2} - \frac{1}{\varepsilon_1^{2}}\right|}.$$

Hence, we obtain the ratio such that

$$\frac{\left\|\alpha^{*}\left(s^{*}\right)M\right\|}{\left\|\alpha(s)M\right\|} : \frac{\left\|\alpha^{*}\left(s^{*}\right)M^{*}\right\|}{\left\|\alpha(s)M^{*}\right\|} = \sqrt{\left|\lambda^{2}\varepsilon_1^{2} - 1\right|}\sqrt{\left|\lambda^{2}\varepsilon_1^{*2} - 1\right|}$$

$$\neq \text{ constant.}$$

Result 3: Mannheim's theorem is invalid for the timelike Bertrand B-pair curves.

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References

[1] Bros, J., Moschella, U., Damour, T., & Darrigol, O.The Geometry of Relativistic Spacetime: from Euclid.s Geometry to Minkowski.s Spacetime. Séminaire Poincaré, 1, (2005) 1-45.
[2] L. R. Bishop, There is more than one way to frame a curve, Amer Math Monthly, vol. 82, (1975), pp.246-251.
[3] S. Yılmaz, M. Turgut, A new version of Bishop frame and an application to spherical images, J. Math. Anal. Appl., vol. 371, (2010), pp. 764-776.
[4] E. Özyılmaz, Classical differential geometry of curves according to type-2 Bishop trihedra, Math.Comput. Appl., vol. 16, no.4, (2011), pp. 858-867.
[5] Y. Ünlütürk, S. Yılmaz, A new version of Bishop frame and its application to Smarandache curves of a spacelike curve in Minkowski 3-space to appear
[6] S. Yılmaz, M. Turgut, A new version of Bishop frame and an application to spherical images, J. Math. Anal. Appl., vol. 371, (2010), pp. 764-776.
[7] S. Yılmaz, Bishop spherical images of a spacelike curve in Minkowski 3-space, Int. Jour. Phys. Scie. vol. 5, no. 6, (2010), pp. 898-905.
[8] S. Yılmaz, Position vectors of some special space-like curves according to Bishop frame in Minkowski space , Sci. Magna., vol. 5, no. 1, (2009), pp. 58-50.
[9] S. Yılmaz, Y. Ünlütürk, A. Mağden, A study on the characterizations of non-null curves according to the Bishop frame of type-2 in Minkowski 3-space, SAÜ Fen Bil Der 20. Cilt, 2. Sayı, (2016), s. 325-335.
[10] F. Yerlikaya, S. Karaahmetoğlu, İ. Aydemir, On the Bertrand B-Pair Curve in 3-Dimensional Euclidean Space, Journal of Science and Arts, 3(36), (2016), 215-224.
[11] Lopes, C.M.C. (2002), Superf´ıcies de Tipo Espa¸co com Vector Curvatura M´edia Nulo em L3 e L4, Master thesis, IME – Universidade de S˜ao Paulo.

[12] O'Neill, B. (1983), *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London.
[13] S. Yılmaz, Y. Ünlütürk, A. Mağden, A study on the characterizations of non-null curves according to the Bishop frame of type-2 in Minkowski 3-space, SAÜ Fen Bil Der 20. Cilt, 2. Sayı, (2016), s. 325-335.