Article

On the Geometry of Surfaces in Spatial Motions

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Abstract

In this study, the rotation motion of a body and its properties is given. Gauss curvature of quadratic surface obtained by this rotation motion is examined.

Keywords: Geometry, surface, spatial motion, rotation motion, Gauss curvature, quadratic surface.

I. INTRODUCTION

The rotation of a three dimensional body M, with respect to a fixed body, F is represented by the transformation equation

$$X = AY \tag{1.1}$$

where X and Y are column matrix which are composed of composite vectors $\boldsymbol{x}$and $\boldsymbol{y}$ respectively. A is an orthogonal matrix . Rotations are orthogonal matrices with determinant equal to 1.

Characteristic polynomial of A is

$$(\lambda - 1)\left(\lambda^2 - \lambda\left(a_{11} + a_{22} + a_{33} - 1\right) + 1\right) = 0.$$

This equation is a reel root $\lambda = 1$. Let b be the eigenvector of A associated with $\lambda = 1$, all points on the line $l = t\boldsymbol{b}$ direction $\boldsymbol{b}$ are fixed during the rotation. This is the axis of rotation of the body [5].

The requirement that a rotation maintain a constant distance between points of the body, thus can be written in the form

$$(X + Y)^T (X + Y) = 0. \tag{1.2}$$

This expresses the fact that diagonals $\boldsymbol{x} - \boldsymbol{y}$ and $\boldsymbol{x} + \boldsymbol{y}$ rhombus with edges $\boldsymbol{x}$ and $\boldsymbol{y}$ intersect at right angles. Now, since $X + Y = (A + I)Y$ and $X - Y = (A - I)Y$, we can compute

$$X - Y = (A - I)(A + I)^{-1}(X + Y). \tag{1.3}$$

From here is

$$B = (A - I)(A + I)^{-1} \tag{1.4}$$

where matrix B is called Cayley's Formula. This the matrix B has the property that $B = -B^T$ which is termed skew-symmetry, that is

$$B = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}.$$

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Those elements can be assemled into the $b = (b_1, b_2, b_3)$ [8].

Given an orthogonal matrix A, we have from (1.3) a skew-symmetric matrix B and the relation

$$X - Y = B(X + Y) \tag{1.5}$$

between the fixed and moving frame coordinates of points in a rotated body. We can write this equation in the form

$$\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{b} \wedge (\boldsymbol{x} + \boldsymbol{y}). \tag{1.6}$$

Equation (1.6) is called Rodrigues' Equation for rotations, $\overrightarrow{b}$ is known as Rodrigues' Vector and norm $\overrightarrow{b}$ is

$$\|\boldsymbol{b}\| = \tan\frac{\phi}{2}. \tag{1.7}$$

Let $\overrightarrow{s} = (s_x, s_y, s_z)$ be the unit vector which is direction vector $\overrightarrow{b}$. Thus is

$$b_1 = \tan\frac{\phi}{2}s_x \ , \ b_2 = \tan\frac{\phi}{2}s_y \ , \ b_3 = \tan\frac{\phi}{2}s_z. \tag{1.8}$$

Cayley's Formula for the orthogonal matrix A can be written in terms of the rotation angle ϕ and the unit vector $\overrightarrow{s}$ by nothing that $B = \tan\frac{\phi}{2}S$. The result is

$$A = (\cos\frac{\phi}{2}I - \sin\frac{\phi}{2}S)^{-1}(\cos\frac{\phi}{2}I + \sin\frac{\phi}{2}S). \tag{1.9}$$

The constants in $C = (\cos\frac{\phi}{2}I + \sin\frac{\phi}{2}S)$ are Euler parameters of A. $(\cos\frac{\phi}{2}I - \sin\frac{\phi}{2}S)^{-1}$ compute the inverse and multiply by C to obtain the expression

$$A = I + \sin\phi S + (1 - \cos\phi)S^2 \tag{1.10}$$

where S is a skew-symmetric matrix [8].

II.QUADRATIC SURFACES AND THEIR CURVATURES

From expressions (1.5) and (1.6) are obtained as follows,

$$\begin{aligned} x_1 - y_1 &= -(x_2 + y_2)b_3 + (x_3 + y_3)b_2 2.1 \\ x_2 - y_2 &= (x_1 + y_1)b_3 - (x_3 + y_3)b_1 \\ x_3 - y_3 &= -(x_1 + y_1)b_2 + (x_2 + y_2)b_2. \end{aligned} \tag{0.1}$$

In addition to we are written quadratic form as follow,

$$f(\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y}) = \sum_{i,j=1}^{n} a_{ij}(x_i - y_i)(x_j + y_j). \tag{2.2}$$

From (2.1) and (2.2) we have as follow in the matrix form

$$g(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y}) = (X + Y)A(X + Y)^T.$$

If we is taken $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{v}$, we can be obtained $g(\boldsymbol{v}, \boldsymbol{v}) = VAV^T$.Thus, the quadrqtic form is obtained as follows,

$$\begin{aligned} g(\boldsymbol{v}) &= (a_{21}b_3 - a_{31}b_2)v_1^2 + (-a_{12}b_3 + a_{32}b_1)v_2^2 2.3 \\ &+(a_{13}b_2 - a_{23}b_1)v_3^2 \\ &+2[\frac{1}{2}(a_{11}b_3 + a_{22}b_3 + a_{31}b_1 - a_{32}b_2)v_1v_2 \\ &+\frac{1}{2}(a_{11}b_2 - a_{21}b_1 + a_{23}b_3 - a_{32}b_2)v_1v_3 \\ &+\frac{1}{2}(a_{12}b_2 + a_{13}b_3 - a_{22}b_1 + a_{23}b_1)v_2v_3] \end{aligned} \tag{0.2}$$

where a_{ij} is computed from (2.2) equation and if we are taken of unit vector $\boldsymbol{s} = (s_x, s_y, s_z)$, we can be obtained quadratic surface as follow,

$$g(\boldsymbol{v}) = \left[\langle \boldsymbol{v}, \boldsymbol{v} \rangle - (s_x v_1 + s_y v_2 + s_z v_3)^2\right] \tan\left(\frac{\Phi}{2}\right) \sin\Phi. \tag{2.4}$$

Now, from equation (2.4), Gauss curvature is obtined as follows,

$$K = \frac{\det(z, D_x z, D_y z)}{\|z\|^4}, \tag{2.5}$$

where is

$$\begin{aligned} z &= \frac{1}{2\tan\frac{\phi}{2}\sin\phi} \bigtriangledown g \\ x &= (\frac{v_2 - (s_x v_1 + s_y v_2 + s_z v_3)s_y}{v_1 - (s_x v_1 + s_y v_2 + s_z v_3)s_z}, 1, 0) \\ y &= (0, \frac{(v_1 - (s_x v_1 + s_y v_2 + s_z v_3)s_x)(v_3 - (s_x v_1 + s_y v_2 + s_z v_3)s_z)}{v_2 - (s_x v_1 + s_y v_2 + s_z v_3)s_y}, \\ &\quad v_1 - (s_x v_1 + s_y v_2 + s_z v_3)s_x). \end{aligned}$$

Consequently, expressions $D_x z$ and $D_y z$ are computed and if they are written equation (2.5), we can be obtained as follows,

$$K = 0.$$

Corollary (2.1): The only surface of revolution with $K = 0$ are the right circular cylinder, the right circular cone, and the plane.

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