Article

Ignorable Variables & Constants of Motion

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Abstract

Torres del Castillo and Rubalcava-García show that each variational symmetry implies the existence of an ignorable coordinate. Here we employ the Lanczos approach to the Noether's theorem to motivate the principal relations of these authors.

Keywords: Variational symmetry, ignorable variable, Noether's theorem, Lanczos variational technique.

1. Introduction

In the functional $S = \int_{t_1}^{t_2} L(q, \dot{q}\,, t)\, dt$ we apply the infinitesimal transformation ($\varepsilon =$ constant $\ll 1$):

$$\tilde{t} = t + \varepsilon\, \xi(q,t), \qquad \tilde{q}_r = q_r + \varepsilon\, \eta_r(q,t), \quad r = 1, \dots, n \tag{1}$$

that is:

$$\tilde{S} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}\,, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t})\, d\tilde{t}\,, \tag{2}$$

then we have a variational symmetry of the Lagrangian if [1]:

$$L\left(\tilde{q}\,, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t}\right) d\tilde{t} = L\left(q, \frac{dq}{dt}, t\right) + \frac{d}{dt}(\varepsilon\, Q), \tag{3}$$

hence the Euler-Lagrange equations [2] corresponding to the variational principle $\delta S = 0$:

$$E_r \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \quad r = 1, \dots, n \tag{4}$$

keep their structure under (1).

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Therefore, we have a symmetry up to divergence and Noether [3-6] proved the existence of the Rund-Trautman identity [7-10]:

$$\frac{\partial L}{\partial q_r}\,\eta_r + \frac{\partial L}{\partial \dot{q}_r}\,\dot{\eta}_r + \frac{\partial L}{\partial t}\,\xi - \left(\frac{\partial L}{\partial \dot{q}_r}\,\dot{q}_r - L\right)\dot{\xi} - \frac{dQ}{dt} = 0, \tag{5}$$

which can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\,\eta_r - H\,\xi - Q\right) = (\eta_r - \dot{q}_r\,\xi)\,E_r\,, \qquad H = \frac{\partial L}{\partial \dot{q}_c}\,\dot{q}_c - L\,. \tag{6}$$

In (5) and (6) we use the convention of Dedekind [11, 12]-Einstein because we sum over repeated indices. If in (6) we employ the Euler-Lagrange equations (4) we deduce the constant of motion associated to (1):

$$\varphi(q,\dot{q}\,,t) \equiv \frac{\partial L}{\partial \dot{q}_r}\,\eta_r - H\,\xi - Q, \tag{7}$$

which establishes the connection between symmetries and conservation laws [13-16].

Now we look a finite transformation of coordinates:

$$t\,,\ q_r \qquad \longrightarrow \qquad t',\ q'_r\,, \qquad r = 1, \ldots, n \tag{8}$$

such that in the new Lagrangian one coordinate, we say q'_1 , participates as ignorable variable and its conjugate momentum leads to the constant (7); Torres del Castillo and Rubalcava-García [17] show that (8) can be obtained from the equations:

$$\frac{\partial t}{\partial q'_1} = \xi\,, \qquad \frac{\partial q_r}{\partial q'_1} = \eta_r\,, \qquad r = 1, \ldots, n\,. \tag{9}$$

The Lanczos variational technique [2, 18-20] allows deduce the Noether's conserved quantity (7) as the Euler-Lagrange equation for the parameter ε if it is considered as a new degree of freedom. Here we employ this Lanczos approach to motivate the relations (9).

2. Torres del Castillo & Rubalcava-García expressions

Lanczos [2, 18] proposes to apply (1) into (2) but considering that ε is a function, therefore up to 1th order in ε:

$$\tilde{S} = \int_{t_1}^{t_2} [L + \dot{\varepsilon}\,\varphi(q,\dot{q},t) + \frac{d}{dt}(\varepsilon Q)]\,dt = \int_{t_1}^{t_2} [\bar{L} + \frac{d}{dt}(\varepsilon Q)]\ dt, \tag{10}$$

where we use (5) and (7); thus we can see that $\varepsilon(t)$ is ignorable into $\bar{L}$ and its corresponding Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\varepsilon}}\right) - \frac{\partial \bar{L}}{\partial \varepsilon} = 0$ implies that φ is a constant.

On the other hand, from (1) we have that $t = \tilde{t} - \varepsilon\, \xi(q,t)$ and $q_r = \tilde{q}_r - \varepsilon\, \eta_r(q,t)$, then:

$$\frac{\partial t}{\partial \varepsilon} = -\,\xi\,, \qquad \frac{\partial q_r}{\partial \varepsilon} = -\,\eta_r\,, \qquad r = 1, \dots, n \tag{11}$$

but ε is ignorable and its momentum leads to (7), hence it is natural the identification $\varepsilon = -\,q_1'$, thus (11) imply the expressions (9) obtained by Torres del Castillo & Rubalcava-García [17], and in their paper we find several examples on the construction of ignorable variables associated to variational symmetries.

Finally, let's remember [2] that the fundamental importance of the ignorable variables for the integration of the Lagrangian equations was first recognized by Routh [21] and Helmholtz [22].

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