Article

Characterization of Quaternionic Curves by Inextenaible Flows

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Abstract. In this article, we investigate flows for quaternionic curves in $\mathbb{E}^4$. We obtain the geometric properties in terms of inextensible flows. Additionally, the characteristic properties for such curves are given.

Keywords: Inextensible flows, Euclidean spaces, quaternion algebra.

1 Introduction

In 1843, quaternions have been proposed by Hamilton who prolonged 3-dimensional linear algebra for inclusion of multiplications or divisions, [6]. Geometrically, quaternions are essentially multi-dimensional complex numbers in space. In view of a standard complex number has a scalar component and an imaginary part, with quaternions the imaginary part is an imaginary vector found on three imaginary orthogonal axes. Additionally, quaternions are both relatively simple and very effective for rotations. So, the quaternion algebra has performed a significant performance lately in some areas of the applied physical science; especially, in differential geometry, in analysis and synthesis of mechanism and machines, image of particle motion in applied molecular physics and the quaternionic equation of motion in theory of general relativity.

It is well known that motion theory has derived a great deal of attention from dynamical systems, biology, mathematical physics, computer vision, and image processing.The problem is interesting since we may set two different subjects on the same theoretical basis. One is a geometrical interpretation of integrable systems. It has been shown that, differential geometry of curve motions is extended to more general types of motion and other integrable systems [1,2,10]. The other is surface dynamics, the dynamics of shapes in physical and biological systems, as in crystal growth.

On the other hand, quaternionic curves have been defined and harmonic curvatures have been obtained. Quaternion valued functions, harmonic curvatures for quaternionic inclined curves have been discussed in the semi-Euclidean space E_2^4 [5]. Quaternionic rectifying curves, Darboux vector of the spatial quaternionic curve and spatial quaternionic Smarandache curves have been proposed by a number of researchers [3-9,11,12].

2 Preliminaries

In this part, we give a brief summary of quaternionic curves to provide the necessary background.

A real quaternion is given by

$$\mathbf{q} = k\mathbf{E}_1 + l\mathbf{E}_2 + m\mathbf{E}_3 + n$$

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where a, b, c, d are constant and

$$\begin{aligned} \mathbf{E}_4 &= 1, \quad \mathbf{E}_1^2 = \mathbf{E}_2^2 = \mathbf{E}_3^2 = -1, \\ \mathbf{E}_1 \times \mathbf{E}_2 &= \mathbf{E}_3, \quad \mathbf{E}_2 \times \mathbf{E}_3 = \mathbf{E}_1, \mathbf{E}_3 \times \mathbf{E}_1 = \mathbf{E}_2, \\ \mathbf{E}_2 \times \mathbf{E}_1 &= -\mathbf{E}_3, \quad \mathbf{E}_3 \times \mathbf{E}_2 = -\mathbf{E}_1, \mathbf{E}_1 \times \mathbf{E}_3 = -\mathbf{E}_2. \end{aligned}$$

If we take $S_q = d$ and $\mathbf{V}_q = k\mathbf{E}_1 + l\mathbf{E}_2 + m\mathbf{E}_3$, a quaternion can be expressed as $\mathbf{q} = S_q + \mathbf{V}_q$. Using this basic products we can now expand the product of two quaternions as

$$\mathbf{p} \times \mathbf{q} = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q \ .$$

There is a unique involutory antiautomorphism of the quaternion algebra [8], denoted by the symbol γ and defined as follows:

$$\gamma \mathbf{q} = -k\mathbf{E}_1 - l\mathbf{E}_2 - m\mathbf{E}_3 + n \ .$$

$\gamma \mathbf{q}$ is called "*Hamiltonian conjugation*". This defines the symmetric,real valued, non-degenerate, bilinear form h as follows:

$$h(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \left[\mathbf{p} \times \gamma \mathbf{q} + \mathbf{q} \times \gamma \mathbf{p} \right] \text{ for } \mathbf{p}, \mathbf{q} \in \mathbf{Q}_H.$$

Also, the norm of $\mathbf{q}$ real quaternion given

$$\|\mathbf{q}\|^2 = h(\mathbf{p}, \mathbf{q}) = \mathbf{q} \times \gamma \ .$$

The concept of a spatial quaternion will be made use of throughout our work. $\mathbf{q}$ is called a spatial quaternion whenever $\mathbf{q} + \gamma \mathbf{q} = 0$.

Let β be a smooth quaternionic curve in $\mathbb{E}^4$ with curvatures $\{K, k, r-K\}$ and $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ denotes the Serret Frenet frame of the quaternionic β. Then Frenet formulas are given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

where K is the principal curvature, k is torsion of β and $(r-K)$ is bitorsion of β [7,8].

3 Geometry of Motion Quaternionic Curves in $\mathbb{E}^4$

The flow of β can be given as

$$\frac{\partial \beta}{\partial t} = \mathfrak{q}_1 \mathbf{T} + \mathfrak{q}_2 \mathbf{N} + \mathfrak{q}_3 \mathbf{B}_1 + \mathfrak{q}_4 \mathbf{B}_2.$$

Putting arclength variation be

$$s(u,t) =_0^u vdu.$$

Definition 3.1. *The flow* $\frac{\partial \beta}{\partial t}$ *are called inextensible if*

$$\frac{\partial}{\partial t} \left| \frac{\partial \beta}{\partial u} \right| = 0.$$

Lemma 3.2. *The flow* $\dfrac{\partial\beta}{\partial t}$ *is inextensible iff*

$$\frac{\partial v}{\partial t} = \frac{\partial \mathfrak{q}_1}{\partial u} + \mathfrak{q}_2 v K.$$

Theorem 3.3. $\dfrac{\partial\beta}{\partial t}$ *is a smooth inextensible flow iff*

$$\frac{\partial \mathfrak{q}_1}{\partial u} = \mathfrak{q}_2 v K.$$

Assume that $v = 1$.

Lemma 3.4. *Let* $\dfrac{\partial\beta}{\partial t}$ *is a smooth inextensible flow. Then,*

$$\begin{aligned}
\frac{\partial \mathbf{T}}{\partial t} &= (\mathfrak{q}_1 K + \frac{\partial \mathfrak{q}_2}{\partial s} - \mathfrak{q}_3 k)\mathbf{N} + (\mathfrak{q}_2 k + \frac{\partial \mathfrak{q}_3}{\partial s} - \mathfrak{q}_4 r + \mathfrak{q}_4 K)\mathbf{B}_1 \\
&\quad + (\mathfrak{q}_3 r - \mathfrak{q}_3 K + \frac{\partial \mathfrak{q}_4}{\partial s})\mathbf{B}_2, \\
\frac{\partial \mathbf{N}}{\partial t} &= -(\mathfrak{q}_1 K + \frac{\partial \mathfrak{q}_2}{\partial s} - \mathfrak{q}_3 k)\mathbf{T} + \psi_1 \mathbf{B}_1 + \psi_2 \mathbf{B}_2, \\
\frac{\partial \mathbf{B}_1}{\partial t} &= (\mathfrak{q}_2 k + \frac{\partial \mathfrak{q}_3}{\partial s} - \mathfrak{q}_4 r + \mathfrak{q}_4 K)\mathbf{T} - \psi_1 \mathbf{N} + \psi_3 \mathbf{B}_2, \\
\frac{\partial \mathbf{B}_2}{\partial t} &= (\mathfrak{q}_3 r - \mathfrak{q}_3 K + \frac{\partial \mathfrak{q}_4}{\partial s})\mathbf{T} - \psi_2 \mathbf{N} - \psi_3 \mathbf{B}_1,
\end{aligned}$$

where

$$\psi_1 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_1 \right\rangle, \ \psi_2 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_2 \right\rangle, \ \psi_3 = \left\langle \mathbf{B}_2, \frac{\partial \mathbf{B}_1}{\partial t} \right\rangle.$$

Proof. Under the assumption, we have

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \beta}{\partial s} = \frac{\partial}{\partial s}(\mathfrak{q}_1 \mathbf{T} + \mathfrak{q}_2 \mathbf{N} + \mathfrak{q}_3 \mathbf{B}_1 + \mathfrak{q}_4 \mathbf{B}_2).$$

Thus, it is seen that

$$\begin{aligned}
\frac{\partial \mathbf{T}}{\partial t} &= (\frac{\partial \mathfrak{q}_1}{\partial s} - \mathfrak{q}_2 K)\mathbf{T} + (\mathfrak{q}_1 K + \frac{\partial \mathfrak{q}_2}{\partial s} - \mathfrak{q}_3 k)\mathbf{N} \\
&\quad + (\mathfrak{q}_2 k + \frac{\partial \mathfrak{q}_3}{\partial s} - \mathfrak{q}_4 (r - K))\mathbf{B}_1 + (\mathfrak{q}_3 (r - K) + \frac{\partial \mathfrak{q}_4}{\partial s})\mathbf{B}_2.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\frac{\partial \mathbf{T}}{\partial t} &= (\mathfrak{q}_1 K + \frac{\partial \mathfrak{q}_2}{\partial s} - \mathfrak{q}_3 k)\mathbf{N} + (\mathfrak{q}_2 k + \frac{\partial \mathfrak{q}_3}{\partial s} - \mathfrak{q}_4 (r - K))\mathbf{B}_1 \\
&\quad + (\mathfrak{q}_3 (r - K) + \frac{\partial \mathfrak{q}_4}{\partial s})\mathbf{B}_2.
\end{aligned}$$

From the above and using

$$\left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{N} \right\rangle = \left\langle \frac{\partial \mathbf{B}_1}{\partial t}, \mathbf{B}_1 \right\rangle = \left\langle \frac{\partial \mathbf{B}_2}{\partial t}, \mathbf{B}_2 \right\rangle = 0,$$

we obtain

$$\begin{aligned}\frac{\partial \mathbf{N}}{\partial t} &= -(\mathfrak{q}_1 K + \frac{\partial \mathfrak{q}_2}{\partial s} - \mathfrak{q}_3 k)\mathbf{T} + \psi_1 \mathbf{B}_1 + \psi_2 \mathbf{B}_2, \\ \frac{\partial \mathbf{B}_1}{\partial t} &= (\mathfrak{q}_2 k + \frac{\partial \mathfrak{q}_3}{\partial s} - \mathfrak{q}_4(r-K))\mathbf{T} - \psi_1 \mathbf{N} + \psi_3 \mathbf{B}_2, \\ \frac{\partial \mathbf{B}_2}{\partial t} &= (\mathfrak{q}_3(r-K) + \frac{\partial \mathfrak{q}_4}{\partial s})\mathbf{T} - \psi_2 \mathbf{N} - \psi_3 \mathbf{B}_1,\end{aligned}$$

where

$$\psi_1 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_1 \right\rangle,\ \psi_2 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_2 \right\rangle,\ \psi_3 = \left\langle \mathbf{B}_2, \frac{\partial \mathbf{B}_1}{\partial t} \right\rangle.$$

Theorem 3.5. *Let* $\frac{\partial \beta}{\partial t}$ *be a smooth inextensible flow of the curve* γ*. Then*

$$\frac{\partial k_1}{\partial t} = \frac{\partial}{\partial s}(\mathfrak{q}_1 K) + \frac{\partial^2 \mathfrak{q}_2}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{q}_3 k) - \mathfrak{q}_2 k^2 - \frac{\partial \mathfrak{q}_3}{\partial s} k + \mathfrak{q}_4 k(r-K).$$

Corollary 3.6.

$$\begin{gathered}\frac{\partial}{\partial s}(\mathfrak{q}_2 k) + \frac{\partial^2 \mathfrak{q}_3}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{q}_4(r-K)) + \psi_1 K = \mathfrak{q}_1 K k + \frac{\partial \mathfrak{q}_2}{\partial s} k - \mathfrak{q}_3 k^2 + \mathfrak{q}_3 (r-K)^2 + \frac{\partial \mathfrak{q}_4}{\partial s}(r-K), \\ \frac{\partial (r-K)}{\partial t} = \psi_2 k + \frac{\partial \psi_3}{\partial s}, \\ \frac{\partial \psi_1}{\partial s} - \psi_2(r-K) = -\mathfrak{q}_2 K k - \frac{\partial \mathfrak{q}_3}{\partial s} K + \mathfrak{q}_4 K(r-K) + \frac{\partial k}{\partial t},\end{gathered}$$

Corollary 3.7. i.

$$\frac{\partial \psi_2}{\partial s} + \psi_1(r-K) = -\mathfrak{q}_3 K(r-K) + \frac{\partial \mathfrak{q}_4}{\partial s} k_1 + \psi_3 k,$$

ii.

$$\mathfrak{q}_2 K k + \frac{\partial \mathfrak{q}_3}{\partial s} K - \mathfrak{q}_4 K(r-K) - \frac{\partial \psi_1}{\partial s} = -\frac{\partial k}{\partial t} - \psi_2(r-K),$$

iii.

$$-\frac{\partial K}{\partial t} + \mathfrak{q}_2 k^2 + \frac{\partial \mathfrak{q}_3}{\partial s} k - \mathfrak{q}_4 k(r-K) = -\frac{\partial}{\partial s}(\mathfrak{q}_1 K) - \frac{\partial^2 \mathfrak{q}_2}{\partial s^2} + \frac{\partial}{\partial s}(\mathfrak{q}_3 k).$$

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