Article

Double Integrals & Riemann Zeta Function

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Abstract

We exhibit elementary proofs of the integral representations of Beukers and Hadjicostas for ζ(*k*), *k* = 2, 3, 4.

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1. Introduction

Beukers [1] obtained the following integral representations for two values of the Riemann zeta function [2, 3]:

$$I_0 \equiv \int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \zeta(2), \qquad I_1 \equiv -\int_0^1 \int_0^1 \frac{Ln\,(xy)}{1-xy}\, dx\, dy = 2\,\zeta(3), \tag{1}$$

which allow to give alternative proofs [4] of the irrationality of ζ(2) and ζ(3) [5, 6].

Hadjicostas [7] deduced the generalization of (1):

$$I_s \equiv \int_0^1 \int_0^1 \frac{[-Ln\,(xy)\,]^{\,s}}{1-xy}\, dx\, dy = \Gamma(s+2)\,\zeta(s+2)\,, \qquad s = 0, 1, 2, 3, \ldots \tag{2}$$

and Guillera-Sondow [8] showed the validity of (2) for complex values of s if Re $s > 1$.

In Sec. 2 we exhibit elementary proofs of $I_n,\ n = 0, 1, 2$, that is, of the integral representations of ζ(*k*), *k* = 2, 3, 4.

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2. Beukers and Hadjicostas formulae

Here first we consider I_0 where a simple integration in *x*, and the use of a Taylor series imply the value indicated in (1):

$$I_0 = \int_0^1 \left[-\frac{1}{y} Ln\,(1-xy)\right]_{x=0}^{x=1} dy = -\int_0^1 \frac{1}{y}\, Ln\,(1-y)\, dy = \int_0^1 \sum_{k=1}^{\infty} \frac{y^{k-1}}{k}\, dy,$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2}\, [y^k]_0^1 = \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2),$$

in according with the result of Beukers. Similarly:

$$I_1 = \sum_{k=1}^{\infty} \int_0^1 \int_0^1 \frac{1}{k}(1-xy)^{k-1}\, dx\, dy\,, \quad \text{because} \quad Ln\,(xy) = -\sum_{k=1}^{\infty} \frac{1}{k}(1-xy)^k,$$

$$= \sum_{k=1}^{\infty} \int_0^1 \frac{1}{k}\left[-\frac{1}{ky}(1-xy)^k\right]_{x=0}^{x=1} dy = \sum_{k=1}^{\infty} \frac{1}{k^2}\int_0^1 \sum_{r=0}^{k-1} (1-y)^{k-r-1}\, dy,$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{r=0}^{k-1} \frac{1}{k-r} = \sum_{k=1}^{\infty} \frac{H_k}{k^2},$$

with the harmonic number $H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$; but we have the identity of Euler (1775) [9]-Stark [10, 11]:

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\,\zeta(3), \qquad (3)$$

then it is immediate the value (1) for I_1.

On the other hand, in [9] we find the relations:

$$[\,Ln\, z\,]^2 = 2 \sum_{k=1}^{\infty} \frac{H_k}{k+1}(1-z)^{k+1}, \qquad \sum_{k=1}^{\infty} \frac{H_k\, H_{k+1}}{(k+1)^2} = 3\,\zeta(4)\,, \qquad (4)$$

which are useful in the following proof:

$$I_2 \equiv \int_0^1 \int_0^1 \frac{[\,Ln\,(xy)\,]^2}{1-xy} dx\, dy = 2\int_0^1 \int_0^1 \sum_{k=1}^{\infty} \frac{H_k}{k+1}(1-xy)^k dx\, dy =$$
$$-2\int_0^1 \sum_{k=1}^{\infty} \frac{H_k[(1-y)^{k+1}-1]}{(k+1)^2\, y}\, dy,$$

$$= 2\int_0^1 \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \sum_{r=0}^{k} (1-y)^{k-r}\, dy = 2\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \sum_{r=0}^{k} \frac{1}{k-r+1} = 2\sum_{k=1}^{\infty} \frac{H_k\, H_{k+1}}{(k+1)^2} = 6\,\zeta(4),$$

in according with (2) for $s = 2$ because $\Gamma(4) = 6$.

It is interesting to indicate that Sondow [12, 13] deduced a similar expression to (1), for the Euler-Mascheroni's constant [14]:

$$\gamma = \int_0^1 \int_0^1 \frac{x-1}{(1-xy)\, Ln\, (xy)}\, dx\, dy, \tag{5}$$

which can be important to establish the possible irrationality of this constant.

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