**Article**

# Biharmonic Curves in Isotropic Space $I_3^1$

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**Abstract.** In this paper, biharmonic curves are studied in Isotropic space $I_3^1$ and in the equiform geometry of Isotropic space $I_3^1$. By using curvature and torsion of the curves, some characterizations are given.

**Keywords:** Biharmonic curve, isotropic space, equiform geometry, isotropic space.

## 1. Introduction

The theory of biharmonic maps is an old and rich subject, initially studied due to its implications in the theory of elasticity and fluid mechanics. G.B. Airy and J.C. Maxwell were the first to study and express plane elastic problems in terms of the biharmonic equation.

The Riemannian generalization of the elastic energy, called the bioenergy, is defined as:

$$E_2(c) = \frac{1}{2}\kappa^2 ds,$$

where $\kappa$ is the geodesic curvature of the curve $c$. Critical points of $E_2$, called biharmonic curves, are described by the equation [12]:

$$\bigtriangledown_c^3 c = R\left(c, \bigtriangledown_c^c\right) c.$$

Biharmonic maps have been extensively studied in the last decade and there are two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

Chen and Ishikawa [2] classified biharmonic curves in semi-Euclidean space $E_v^n$. They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Thus, it suffices to classify biharmonic curves in semi-Euclidean 3-space. More recently, besides in semi-Euclidean space, many studies have been made in other space: In [3], Degla and et al. proved that there was no non-geodesic biharmonic curve in a four-dimensional Damek-Ricci space although such curves exist in three-dimensional Heisenberg groups. Inoguchi gave a differential geometric interpretation for the classification of biharmonic curves in semi-Euclidean 3-space, in [5]. In [6], Kocayiğit and et al. studied 1-type curves by using the mean curvature vector field of the curve and they also studied biharmonic curves whose mean curvature vector field was in the kernel of Laplacian. In [7], Kocayiğit et al. gave definitions and characterizations of harmonic 1-type and weak biharmonic curves by using the mean curvature vector field of a Frenet curve in the Lorentzian 3-space $L^3$.In [8], Körpınar and et al. gave some characterizations by using the curvature and torsion of curves in the $H^2 \times R$. In [10], they found curvature characterizations of biharmonic Legendre curves in S-space forms. In [11], Perktaş and et al. studied the non-geodesic non-null biharmonic in 3-dimensional hyperbolic Heisenberg group with a semi-Riemannian metric of index 2. In [12], Voicu studied biharmonic curves in spaces with Finslerian geometry. He gave specific properties and existence of non-geodesic biharmonic curves for some classes of Finsler spaces.

In this paper, 1-type curves and biharmonic curves are given by using the curvature vector field in Isotropic Space $I_3^1$ and in the equiform geometry of the Isotropic Space $I_3^1$. In Isotropic Space $I_3^1$, we showed that biharmonic curve holds if and only if a Frenet curve is a geodesic. Additionally, in the equiform geometry of Isotropic Space $I_3^1$, we showed that biharmonic curve holds if and only if a Frenet curve is a geodesic and a null cubic.

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**2. Basic notions and properties**

The isotropic geometry is one of the real Cayley-Klein geometries. The absolute of the simple isotropic geometry is an ordered triple $\{w,\ f_1,\ f_2\}$ where $w$ is the ideal (absolute) plane and $f_1,\ f_2$ couple of complex conjugate lines in $w$.

Let $c : I \to I_3^1$, $I \subset IR$ be a curve given by

$$c(s) = (x(s), y(s), z(s)),$$

where $x(s)$, $y(s)$, $z(s) \in C^3$ (the set of three times continuously differentiable functions) and $s$ run through a real interval [9].

Let $c$ be an admissible curve in $I_3^1$, parameterized by arc length, given in coordinate form

$$c(s) = (s, y(s), z(s)). \tag{2.1}$$

Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\begin{aligned} \kappa(s) &= x^{'}y^{''} - y^{'}x^{''} \\ \tau(s) &= \frac{\det(c^{'}(s), c^{''}(s), c^{'''}(s))}{\kappa^2(s)} \end{aligned} \tag{2.2}$$

and associated moving trihedron is given by

$$\begin{aligned} t(s) &= c^{'}(s) \\ n(s) &= \frac{1}{\kappa(s)}c^{''}(s) \\ b(s) &= (0, 0, 1) \end{aligned} \tag{2.3}$$

The vectors $t_c$, $n_c$, $b_c$ are called the vectors of the tangent, principal normal and binormal line of $c$, respectively. For their derivatives the following Frenet formulas hold [9]

$$\begin{aligned} t^{'}(s) &= \kappa(s)n(s) \\ n^{'}(s) &= -\kappa(s)t(s) + \tau(s)b(s) \\ b^{'}(s) &= 0. \end{aligned} \tag{2.4}$$

Scalar product in the Isotropic space $I_3^1$ is defined as follows, if $x_1, x_2 \neq 0$ or $y_1, y_2 \neq 0$, then

$$< X, Y >= x_1y_1 + x_2y_2, \tag{2.5}$$

and if $x_3 = 0$ or $y_3 = 0$, then

$$< X, Y >= x_3y_3, \tag{2.6}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ .

The equiform differential geometry of the Isotropic space $I_3^1$ has been studied in [4]. Let us recall some basic definitions from this work.

The equiform curvature and the equiform torsion of an admissible curve is defined by

$$\mathcal{K} = \dot{\rho}, \qquad \mathcal{T} = \rho\tau = \frac{\tau}{\kappa}, \tag{2.7}$$

where $\rho$ is the radius of curvature of the curve c.
The associated trihedron is given by

$$T = \rho.t, \quad N = \rho.n, \quad B = \rho.b. \tag{2.8}$$

The formulas analogous to the Frenet's in the equiform geometry of the simple isotropic space have the following form

$$\begin{aligned} \frac{dT}{d\sigma} &= \mathcal{K}.T + N, \\ \frac{dN}{d\sigma} &= -T + \mathcal{K}.N + \mathcal{T}.B, \\ \frac{dB}{d\sigma} &= \mathcal{K}.B, \end{aligned} \tag{2.9}$$

where $\sigma$ is an equiform invariant parameter defined by $\sigma = \frac{ds}{\rho}$.

## 3. 1-Type Curves and Biharmonic Curves

Let $c : I \to I_3^1$, $I \subset IR$ be an arclengthed curve in Isotropic space $I_3^1$. Namely the velocity vector field $c'$ satisfies $\langle c', c' \rangle = 1$. A unit speed curve $c$ is said to be a geodesic if $\bigtriangledown_c c = 0$, where $\bigtriangledown$ is the Levi-Civita connection. In particular an arclengthed curve $c$ is said to be a geodesic if $\kappa = 0$, where $\kappa$ is the curvature of $c$. Note that if $\kappa = 0$, then automatically $\tau = 0$, where $\tau$ is the torsion of $c$.
We first assume that $\langle c'', c'' \rangle \neq 0$. A unit speed curve $c$ is said to be a Frenet curve if $\langle c'', c'' \rangle \neq 0$.
Let us denote the Laplace-Beltrami operator by $\triangle$ of $c$ and the mean curvature vector field along $c$ by $H$, [1].
The Frenet-Serret formulae of $c$ in Isotropic space $I_3^1$ imply that the mean curvature vector field $H$ is given by

$$H = \bigtriangledown_{c'} c' = \bigtriangledown_{c'} t = \kappa n, \tag{3.1}$$

where $\kappa$ is the curvature of $c$.
The Laplacian operator of $c$ is defined by

$$\triangle = - \bigtriangledown_{c'}^2 c' = - \bigtriangledown_{c'} \bigtriangledown_{c'} c'. \tag{3.2}$$

**Definition 3.1.** Let $M \subset E^{n+d}$ be a compact submanifold and $x : M \to E^{n+d}$ be an isometric immersion. In the case of

$$x = x_0 +_{i=1}^{k} x_i$$

then $M$ is called finite type where $x_0$ is the constant vector and $\triangle x_i = \lambda_i x_i$, in the other case $M$ is called infinite type and $x_1, x_2, ..., x_k$ are non-constant functions [2,6].
**Theorem 3.1.** The submanifold $M \subset E^{n+d}$ is k-type if and only if the mean curvature vector field $H$ of $M$ satisfy

$$\triangle^k H + c_1 \triangle^{k-1} H + .... + c_{k-1} \triangle H + c_k H = 0,$$

where

$$c_1 = -_{t=p}^{q} \lambda_t$$

$$c_2 = -_{t=p}^{q} \lambda_t \lambda_s, ..., c_{q-p+1} \lambda_{p...} \lambda_q \qquad (k = q - p + 1),$$

where $\triangle x_i = \lambda_i x_i \ \ (1 \leq i \leq k)$, [2,6].
According to the above theorem, the following definition can be given.

**Definition 3.2.** A unit speed curve $c : I \to I_3^1$, $I \subset IR$ in Isotropic space $I_3^1$ is said to be 1-type if

$$\triangle H = \lambda H. \tag{3.3}$$

**Definition 3.3.** A unit speed curve $c : I \to I_3^1$, $I \subset IR$ in Isotropic space $I_3^1$ is said to be biharmonic if

$$\triangle H = 0. \tag{3.4}$$

**Theorem 3.2.** If $c$ is biharmonic curve if and only if $\triangle(\triangle c) = 0$, [6].
**Lemma 3.1.** The mean curvature vector field $H$ is harmonic ($\triangle H = 0$) if and only if [6]

$$\nabla_{c'} \nabla_{c'} \nabla_{c'} c' = 0.$$

**Theorem 3.3.** Let $c$ be an admissible Frenet curve parameterized by the arc of length in Isotropic Space $I_3^1$. Then, the curve $c$ is 1-type if and only if

$$\begin{aligned} \kappa\kappa' &= 0, \\ \kappa'' - \kappa^3 &= \lambda\kappa, \\ 2\kappa'\tau + \kappa\tau' &= 0. \end{aligned} \tag{3.5}$$

**Proof.** From (2.4), (3.1) and (3.2), we have

$$\triangle H = (-3\kappa\kappa')\, t + \left(\kappa'' - \kappa^3\right) n + (2\kappa'\tau + \kappa\tau')\, b. \tag{3.6}$$

By (3.1) and (3.3), we get

$$(3\kappa\kappa')\, t - \left(\kappa'' - \kappa^3\right) n - (2\kappa'\tau + \kappa\tau')\, b = \lambda\kappa n. \tag{3.7}$$

From (3.7), we obtain (3.5). Conversely, the equations of (3.5) satisfy the equation (3.3).
**Theorem 3.4.** Let $c$ be an admissible Frenet curve parameterized by the arc of length in Isotropic Space $I_3^1$. Then the curve $c$ is biharmonic if and only if $c$ is a geodesic.
**Proof.** Let $I$ be an open interval and $c : I \to I_3^1$ be an admissible Frenet curve parameterized by the arc of length. Let $\{t,\ n,\ b\}$ be the Frenet frame field of $c$. Considering (3.1),

$$\nabla_{c'} H = -\kappa^2 t + \kappa' n + \kappa\tau b.$$

Let us compute the Laplacian of H:

$$\begin{aligned} -\triangle H &= \nabla_{c'} \nabla_{c'} H \\ &= (-3\kappa\kappa')\, t + \left(\kappa'' - \kappa^3\right) n + (2\kappa'\tau + \kappa\tau')\, b + \end{aligned}$$

Hence along the curve $c$, $\triangle H = 0$ holds if and only if $\kappa = 0$. So $c$ is a geodesic.
Conversely, every geodesic curve satisfies $\triangle H = 0$.

By a similar calculation, we obtain the following theorems in the equiform geometry of Isotropic Space $I_3^1$.
**Theorem 3.5.** Let $c$ be an admissible Frenet curve parameterized by the arc of length in the equiform geometry of Isotropic Space $I_3^1$. Then the curve $c$ is 1-type if and only if

$$\begin{aligned} \mathcal{K}'' + 3\mathcal{K}\mathcal{K}' + \mathcal{K}^3 - 3\mathcal{K} &= \lambda\mathcal{K} \\ 3\mathcal{K}' + 3\mathcal{K}^2 - 1 &= \lambda \\ 3\mathcal{K}\mathcal{T} + \mathcal{T}\,' &= 0. \end{aligned}$$

**Theorem 3.6.** Let $c$ be an admissible Frenet curve parameterized by the arc of length in the equiform geometry of Isotropic Space $I_3^1$.Then the curve $c$ is biharmonic if and only if $\mathcal{K} = 0$ and $\mathcal{T} = 0$.

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**References**

[1] Chen, B.Y., Decu, S., Verstraelen, L., Notes on Isotropic Geometry of Production Models, arXiv:1401.5415v1[math.OC] 21January 2014.
[2] Chen, B.Y., Ishikava, S., Biharmonic surfaces in pseudo- Euclidean spaces, Man. Fac. Sci. Kyushu Univ. Ser. A 45, no.2 (1991), 323-347.
[3] Degla, S., Todjihounde, L., Biharmonic Curves in Four-Dimensional Damek-Ricci Spaces, Journal of Mathematical Sciences: Advances and Applications, 5(1) (2010), 19-27.
[4] Erjavec, Z., Divjak B., Harvat, D., The General Solutions of Frenet's System in the Equiform Geometry of the Galilean, Pseudo-Galilean , Simple Isotropic and Double Isotropic Space, International Mathematical Forum, Vol.6, no.17 (2011), 837-856.
[5] Inoguchi, J., Biharmonic Curves in Minkowski 3-Space, IJMMS, 21, 1365-1368, Hindawi Publishing Corp., 2003.
[6] Kocayiğit, H., Hacısalihoğlu, H. H., 1-Type curves and Biharmonic curves in Euclidean 3-space, International Electronic Journal of Geometry, Vol.4, No.1 (2011), 97-101.
[7] Kocayiğit, H., Önder, M., Hacısalihoğlu, H. H., Harmonic 1-Type curves and weak Biharmonic curves in Lorentzian 3- space, Analele Stiintifice Ak Univ. tatli Matematica, Tomul Lx, f.1., 2014.
[8] Körpınar, T., Turhan, E., Biharmonic Curves in $H^2 \times R$, Gen. Math. Notes, 4(1) (2011), 59-66.
[9] Öğrenmiş, A.O., Külahcı, M., Bektaş, M., A survey for some special curves in Isotropic Space $I_3^1$, Physical review and Research International, 3(4) (2013), 321-329.
[10] Özgür, C., Güvenç, Ş., On Biharmonic Legendre Curves in S-space Forms, Turk J. Math. 38 (2014), 454-461.
[11] Perktaş, S.Y., Kılıç, E., On Biharmonic Curves in 3-dimensional Heisenberg Group, Adiyaman Univ., Journal Of Science 2(2) (2012), 58-74.
[12] Voicu, N., Biharmonic Curves in Finsler Spaces, J. Korean Math. Soc. 51, No.6, pp. 1105-1122, 2014.