Article

Kinnersley's Metrics

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Abstract

We study the explicit form of each Kinnersley's metric to show that there always exists a Newman-Penrose tetrad generating certain relations between the corresponding spin coefficients; our results imply some identities obtained by Czapor-McLenaghan.

Keywords: Type D empty spacetimes, Newman-Penrose formalism.

1. Introduction

Here we employ the notation, quantities and conventions of [1-6]. Kinnersley [7-9] gave the explicit structure of any solution of Einstein's field equations for vacuum spacetimes of type D in the Petrov classification [4, 10]. We apply the Newman-Penrose (NP) formalism [1-6, 8, 10, 11] to each Kinnersley's metric, taking the null vectors l^α and n^α aligned with the 2-degenerate principal directions [5], then the Goldberg-Sachs theorem [6, 10, 12] implies:

$$\kappa = \sigma = \nu = \lambda = 0, \tag{1}$$

and we show that always it is possible to select the NP tetrad such that:

$$\tau = \pi, \quad \alpha = \beta, \quad \mu = q\,\rho, \quad \gamma = q\,\varepsilon, \quad \psi_2 = 4(\gamma\,\rho - \pi\,\beta), \qquad q = \pm 1. \tag{2}$$

Hence from (2) are immediate the identities of Czapor-McLenaghan [11, 13, 14]:

$$\pi\,\bar{\pi} - \tau\,\bar{\tau} = \rho\,\bar{\mu} - \mu\,\bar{\rho} = 0, \tag{3}$$

and the proportionality of two of their invariants:

$$I_3 \equiv \rho\,\bar{\pi} - \tau\,\bar{\rho}\,, \qquad I_4 \equiv \mu\,\bar{\pi} - \tau\,\bar{\mu} = q\,I_3\,; \tag{4}$$

besides, it shall be clear that $I_3 = I_4 = 0$ for all Kinnersley solutions except for the Class III.

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2. Type D vacuum metrics

For each metric [7-9] we indicate the null tetrad and its corresponding spin coefficients [3] verifying the properties (2); let's remember that $\psi_c = 0,\ c \neq 2$ and $(x^\nu) = (u, r, x, y),\ \nu = 0, \dots, 3$.

1). Class I: This case leads to the NUT geometries [8, 15].

$$(l^\nu) = (0,1,0,0), \quad (n^\nu) = \left(1, -\frac{\mu}{\rho}, 0, 0\right), \quad (m^\nu) = \left(-\frac{ia}{\sqrt{2}}\,\bar{\rho}\,\eta,\ 0,\ \bar{\rho}\,P,\ i\,\bar{\rho}\,P\right), \tag{5}$$

where a is a positive real constant and:

$$\rho = -\frac{1}{r + i\,a}\,, \quad \mu = \mu_0\,\bar{\rho} + \frac{b}{2}(\rho + \bar{\rho})\,\rho, \quad b = b_0 + 2ia\,\mu_0\,, \quad 2\,\mu_0 = 0, \pm 1, \quad \eta = x + iy,$$

$$\gamma = \frac{b}{2}\rho^2\,,\ \alpha = -\bar{\beta} = \frac{\mu_0}{2\sqrt{2}}\,\rho\,\bar{\eta}\,,\ P = \frac{1}{\sqrt{2}}\left(1 + \frac{\mu_0}{2}\eta\,\bar{\eta}\right),\ \psi_2 = b\,\rho^3 = 2\gamma\,\rho,\ \varepsilon = \tau = \pi = 0; \tag{6}$$

it is easy to prove that μ/ρ is real, in fact:

$$\frac{\mu}{\rho} = \mu_0 + \frac{1}{2}\left(b\,\rho + \bar{b}\,\bar{\varrho}\right) = \frac{1}{r^2 + a^2}\left[\mu_0(r^2 - a^2) - b_0 r\right]. \tag{7}$$

In [3, 8] are the following expressions for a general rotation of type III:

$$\tilde{l}^\nu = e^{A}\,l^\nu\,, \quad \tilde{n}^\nu = e^{-A}\,n^\nu\,, \quad \tilde{m}^\nu = e^{-iB}\,m^\nu\,, \quad \tilde{\bar{m}}^\nu = e^{iB}\,\bar{m}^\nu\,,$$

$$\tilde{\kappa} = e^{2A - iB}\kappa,\ \tilde{\rho} = e^{A}\rho,\ \tilde{\sigma} = e^{A-2iB}\sigma,\ \tilde{\tau} = e^{-iB}\tau,\ \tilde{\pi} = e^{iB}\pi,\ \tilde{\nu} = e^{-2A+iB}\nu,\ \tilde{\mu} = e^{-A}\mu,$$

$$\tilde{\lambda} = e^{-A+2iB}\lambda, \quad \tilde{\varepsilon} = e^{A}\left[\frac{1}{2}D(A - iB) + \varepsilon\right], \quad \tilde{\gamma} = e^{-A}\left[\frac{1}{2}\Delta(A - iB) + \gamma\right], \tag{8}$$

$$\tilde{\alpha} = e^{iB}\left[\frac{1}{2}\bar{\delta}(A - iB) + \alpha\right], \quad \tilde{\beta} = e^{-iB}\left[\frac{1}{2}\delta(A - iB) + \beta\right], \quad \tilde{\psi}_2 = \psi_2\,,$$

whose application to (5) and (6) with $A = Ln\sqrt{\frac{\mu}{\rho}}$ and $B = arc\ tg\ \frac{y}{x} - arc\ tg\ \frac{r}{a}$ implies (2) for $q = 1$:

$$\tilde{\tau} = \tilde{\pi} = 0,\ \tilde{\alpha} = \tilde{\beta} = -\frac{i}{2}\sqrt{\frac{\varrho\,\bar{\varrho}}{2\eta\,\bar{\eta}}\left(1 - \frac{\mu_0}{2}\eta\,\bar{\eta}\right)}\,,\ \tilde{\mu} = \tilde{\rho} = \sqrt{\mu\,\rho}\,,\ \tilde{\gamma} = \tilde{\varepsilon} = \frac{\gamma}{2}\sqrt{\frac{\rho}{\mu}}\,, \quad \tilde{\psi}_2 = 4\tilde{\gamma}\,\tilde{\rho}.$$

(9)

2). Class II.A: Kerr-NUT spacetime [16].

$$(l^\nu) = \sqrt{-\frac{qN}{2\Sigma}}\,(0,1,0,0),\ (n^\nu) = \frac{1}{\sqrt{-2qN\Sigma}}(2(r^2+l^2+a^2),\ N,0,2a),\ q=\pm 1,\ -qN>0,$$

(10)

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}(a\sin x + 2l\cot x, 0, -i, \csc x),\ Q = r + i(l - a\cos x),\ N = -r^2 + 2\,m\,r + l^2 - a^2,\ \Sigma = Q\bar{Q},$$

verifying (2) with:

$$\rho = -\frac{1}{Q}\sqrt{-\frac{qN}{2\Sigma}}\,, \qquad \tau = -\frac{a}{Q\sqrt{2\Sigma}}\sin x, \qquad \alpha = \frac{-a+(l-ir)\cos x}{2Q\sqrt{2\Sigma}}\csc x\,,$$

(11)

$$\varepsilon = -\frac{q}{2Q\sqrt{-2qN\Sigma}}\,[a^2 - mr - l^2 + i(m-r)(l - a\cos x)], \qquad \psi_2 = -\frac{m+i\,l}{Q^3}.$$

For the cases $l = a = 0$ & $l = 0$ we obtain the Schwarzschild and Kerr [8, 10, 17] metrics, respectively.

3). Class II.B:

$$(l^\nu) = \sqrt{-\frac{qN}{2\Sigma}}\,(0,1,0,0), (n^\nu) = \frac{1}{\sqrt{-2qN\Sigma}}(2(r^2+l^2+a^2),\ N,0,2a), Q = r + i(a\cosh x - l),$$

(12)

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}(-a\sinh x + 2l\coth x, 0, -i, \operatorname{csch} x),\quad \Sigma = Q\bar{Q},\quad N = r^2 + 2mr - l^2 + a^2,$$

in according with (2) such that:

$$\rho = -\frac{1}{Q}\sqrt{-\frac{qN}{2\Sigma}}\,, \qquad \tau = -\frac{a}{Q\sqrt{2\Sigma}}\sinh x, \qquad \alpha = \frac{a-(l+ir)\cos h\, x}{2Q\sqrt{2\Sigma}}\operatorname{csch} x\,,$$

(13)

$$\varepsilon = -\frac{q}{2Q\sqrt{-2qN\Sigma}}\,[-a^2 - mr + l^2 + i(m+r)(-l + a\cosh x)], \qquad \psi_2 = -\frac{m+i\,l}{Q^3}.$$

4). Class II.C:

$$(l^\nu) = \sqrt{-\frac{qN}{2\Sigma}}\,(0,1,0,0), (n^\nu) = \frac{1}{\sqrt{-2qN\Sigma}}(2(r^2+l^2-a^2),\ N,0,2a), Q = r + i(a\sinh x - l),$$

(14)

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}(-a\cosh x + 2l\tanh x, 0, -i, \mathrm{sech}\, x), \quad \Sigma = Q\bar{Q}, N = r^2 + 2mr - l^2 - a^2,$$

satisfying (2) with:

$$\rho = -\frac{1}{Q}\sqrt{-\frac{qN}{2\Sigma}}\,, \qquad \tau = -\frac{a}{Q\sqrt{2\Sigma}}\cosh x, \qquad \alpha = -\frac{a + (l + ir)\sin h\, x}{2Q\sqrt{2\Sigma}}\,\mathrm{sech}\, x\,,$$

$$\varepsilon = -\frac{q}{2Q\sqrt{-2qN\Sigma}}\,[a^2 - mr + l^2 + i(m + r)(-l + a\sinh x)], \qquad \psi_2 = -\frac{m + i\,l}{Q^3}\,. \tag{15}$$

5). <u>Class II.D</u>:

$$(l^\nu) = \sqrt{-\frac{qN}{2\Sigma}}\,(0,1,0,0), \quad (n^\nu) = \frac{1}{\sqrt{-2qN\Sigma}}(2(r^2 + l^2),\ N, 0, 2a), \quad Q = r + i(a\,e^x - l),$$

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}(-a\,e^x + 2l, 0, -i, e^{-x}), \qquad \Sigma = Q\bar{Q}, \qquad N = r^2 + 2mr - l^2, \tag{14}$$

and (2) are correct with:

$$\rho = -\frac{1}{Q}\sqrt{-\frac{qN}{2\Sigma}}\,, \qquad \tau = -\frac{a\,e^x}{Q\sqrt{2\Sigma}}\,, \qquad \alpha = -\frac{l + i\,r}{2Q\sqrt{2\Sigma}}\,,$$

$$\varepsilon = -\frac{q}{2Q\sqrt{-2qN\Sigma}}\,[-mr + l^2 + i(m + r)(-l + a\,e^x), \qquad \psi_2 = -\frac{m + i\,l}{Q^3}\,. \tag{15}$$

6). <u>Class II.E</u>:

$$(l^\nu) = \sqrt{-\frac{qN}{2\Sigma}}\,(0,1,0,0), \quad (n^\nu) = \frac{1}{\sqrt{-2qN\Sigma}}(2(r^2 + b^2),\ N, 0, 2), \quad Q = r + i\left(b + \frac{x^2}{2}\right),$$

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}\left(-b\,x - \frac{x^3}{4}, 0, -i,\ \frac{1}{x}\right), \qquad \Sigma = Q\bar{Q}, \qquad N = 2(m\,r + b), \quad q = -1, \tag{16}$$

with the properties (2) for:

$$\rho = -\frac{1}{Q}\sqrt{-\frac{qN}{2\Sigma}}\,, \quad \tau = -\frac{x}{Q\sqrt{2\Sigma}}\,, \quad \alpha = \frac{2b - x^2 - 2i\,r}{4\,x\,Q\sqrt{2\Sigma}}\,,$$

$$\varepsilon = \frac{q}{2Q\sqrt{-2qN\Sigma}}\,(m\bar{Q} + 2b), \quad \psi_2 = -\frac{m + i}{Q^3}\,. \tag{17}$$

7). Class II.F:

$$(l^\nu) = \sqrt{-\frac{qN}{2\Sigma}}\ (0,1,0,0), \qquad (n^\nu) = \frac{1}{\sqrt{-2qN\Sigma}}(2r^2,\ N,0,2), \qquad Q = r + i\,x,$$

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}(-x^2,0,-i,\ 1), \qquad \Sigma = Q\bar{Q}, \qquad N = 2mr - 1, \tag{18}$$

in according with (2) for:

$$\rho = -\frac{1}{Q}\sqrt{-\frac{qN}{2\Sigma}}\,, \qquad \tau = 2\alpha = -\frac{1}{Q\sqrt{2\Sigma}}\,,$$

$$\varepsilon = -\frac{q}{2Q\sqrt{-2qN\Sigma}}\ (1 - m\,r + i\,m\,x)), \quad \psi_2 = -\frac{m}{Q^3}\,. \tag{19}$$

8). Class III.A: Static 'C' metric [10, 18, 19].

$$ds^2 = (x+y)^{-2}(h\,du^2 - f\,dr^2 - f^{-1}dx^2 - h^{-1}dy^2),$$

$$f = x^3 + a\,x + b, \quad h = y^3 + a\,y - b, \qquad a,\ b \ \text{are constants,} \tag{20}$$

then we employ the null tetrad:

$$(l^\nu) = \frac{x+y}{\sqrt{2}}\left(\frac{1}{\sqrt{h}},0,0,\sqrt{h}\right), (n^\nu) = \frac{x+y}{\sqrt{2}}\left(\frac{1}{\sqrt{h}},0,0,-\sqrt{h}\right),\ (m^\nu) = \frac{x+y}{\sqrt{2}}\left(0,\frac{1}{\sqrt{f}},-i\sqrt{f},0\right), \tag{21}$$

verifying (2) with $q = 1$ and $\psi_2 = \frac{1}{2}(x+y)^3$:

$$\rho = \sqrt{\frac{h}{2}}\,,\ \ \tau = -i\sqrt{\frac{f}{2}}\,,\ \ \alpha = \frac{i}{4\sqrt{2f}}[2f - (3x^2+a)(x+y)]\,,$$

$$\varepsilon = -\frac{1}{4\sqrt{2h}}[2h - (3y^2+a)(x+y)]\,. \tag{22}$$

We can observe that in this case the invariants (4) are nonzero because $I_3 = i\sqrt{hf}$.

9). Class III.B:

$$(l^\nu) = \sqrt{qE}\ (0,1,0,0), \quad (n^\nu) = \frac{1}{\sqrt{qE}}(X^0,U,0,X^3), \quad q = \pm 1,$$

$$(m^\nu) = \frac{1}{\sqrt{2\Sigma}}\left(\frac{sn\,x}{\pi_0}\ ,-i\sqrt{2}\,\pi_0(r^2+3\rho_0^2),-4a\,\pi_0,\frac{dn\,x}{\pi_0}\right), \quad qE > 0, \tag{23}$$

such that:

$$\Sigma = Q\bar{Q}, \quad Q = r + i\,\rho_0, \quad \rho_0 = a\,cn\,x, \quad \rho = -\frac{1}{Q},$$

$$X^0 = dn\,x + \frac{a\sqrt{2}}{\Sigma}\left(r\,cn\,x + a\sqrt{2}\,sn\,x\,dn\,x\right) sn\,x,$$

$$\pi_0^2 = c\,sn\,x\,dn\,x + \frac{b}{4a^2}\,cn^2x - \frac{\sqrt{2}}{8a^3}\,cn\,x\left(m\,sn\,x + l\sqrt{2}\,dn\,x\right), \mu_0 = \frac{1}{t_0}(2\rho_0^3\,\pi_0^2 - 2\rho_0 U_0 - l_0)$$

$$l_0 = l dn\,x(1 - 2sn^2x) - \frac{m}{\sqrt{2}} sn\,x(3 - 2sn^2x), X^3 = sn\,x - \frac{2a\sqrt{2}}{\Sigma}\left(rcn\,x + a\sqrt{2}sn\,x\,dn\,x\right)dn\,x, \quad (24)$$

$$E = \frac{1}{\Sigma}\left[\tau\,\bar{\tau}_0 - \frac{1}{2}\left(\bar{Q}\,\tilde{\psi}_0 + Q\,\bar{\tilde{\psi}}_0\right) + 2\rho\,\pi_0^2(\rho_0\,r^2 - 2\,r\,t_0 - 3\rho_0^3)\right] + (2\rho_0^2 - r^2)\,\pi_0^2 - \mu_0\,r - U_0\,,$$

$$U_0 = b - 3\rho_0^2\pi_0^2 + \frac{3}{2a\sqrt{2}}\,cn\,x\left(\,m\,sn\,x - l\sqrt{2}\,dn\,x\right), \quad \tilde{\psi}_0 = (m + i\,l)\,(dn\,x - \frac{i}{\sqrt{2}}\,sn\,x)^3,$$

$$t_0 = 2a^2\sqrt{2}\,sn\,x\,dn\,x\,, \qquad a, b, c, l, m \text{ are arbitrary constants,}$$

where $sn\,x, cn\,x, dn\,x$ are elliptic functions of modulus $\frac{1}{\sqrt{2}}$; in [7, 8] we can find the corresponding expression for U. The relations (2) are satisfied, for example:

$$\rho = q\,\mu = -\frac{\sqrt{qE}}{Q}\,, \quad \tau = \pi = \frac{\pi_0}{Q\sqrt{\Sigma}}[t_0 - 2\rho_0\,r + i(r^2 + \rho_0)], \quad \psi_2 = -\frac{\tilde{\psi}_0}{Q^3}, \text{etc.} \quad (25)$$

Plebañski-Demiañski [9, 20, 21] already showed Class III.B could be reduced to Class III.A.

10). <u>Class IV.A</u>:

$$(l^\nu) = (0, 1, 0, 0), \quad (n^\nu) = \left(1, -\frac{l}{2a}\,\frac{r^2}{x^2+a^2}, 0, \frac{4a\,r}{x^2+a^2}\right), \quad (m^\nu) = \left(0, \frac{2rx\xi}{x^2+a^2}, \xi, \frac{i}{\xi}\right), \quad (26)$$

where $\xi^2 = \frac{2am\,x + l\,(a^2 - x^2)}{2a\,(x^2+a^2)}$, therefore:

$$\rho = \mu = \varepsilon = 0, \quad \tau = -\pi = -\frac{x - i\,a}{x^2+a^2}\,\xi\,, \quad \gamma = \frac{l\,r}{2a\,(x^2+a^2)}\,, \quad \alpha = \frac{(l\,x - m\,a) - i\,2\,a^2\,\xi}{4a\,\xi\,(x^2+a^2)}\,,$$

$$\beta = \frac{m\,a - (l + 4a\,\xi^2)\,x + i\,2a^2\xi^2}{4a\,\xi(x^2 + a^2)}\,, \qquad \psi_2 = \frac{m + i\,l}{(x + i\,a)^3} = 2\,\pi\,(\alpha - \beta). \quad (27)$$

Now for (26) and (27) we apply the relations (8), that is, a rotation of type III with $A = Ln[r\sqrt{\frac{el}{2a(x^2+a^2)}}]$ and $B = 90°$, $e = \pm 1$, $\frac{e\,a}{l} > 0$, then (2) are verified for $q = 1$:

$$\tilde{\rho} = \tilde{\mu} = 0, \quad \tilde{\tau} = \frac{(a+i\,x)\,\xi}{x^2+a^2}\,, \quad \tilde{\varepsilon} = \frac{1}{2}\sqrt{\frac{e\,l}{2a\,(x^2+a^2)}}\,, \quad \tilde{\alpha} = i\,\left[\alpha + \frac{x\,\xi}{2(x^2+a^2)}\right], \quad \tilde{\psi}_2 = \psi_2 = -4\tilde{\pi}\,\tilde{\beta}. \tag{28}$$

11). <u>Class IV.B</u>:

$$(l^\nu) = (0,1,0,0), (n^\nu) = \left(1, \frac{c\,r^2}{x^2}, 0, 0\right), (m^\nu) = \left(0,\ \frac{2r\xi}{x},\ \xi,\ \frac{i}{\xi}\right), \quad \xi^2 = c + \frac{m}{x}\,,\ c = 0,\ \pm\frac{1}{2}, \tag{29}$$

hence:

$$\rho = \mu = \varepsilon = 0, \quad \tau = -\pi = \frac{\xi}{x}, \quad \gamma = -\frac{c\,r}{x^2}, \quad \alpha = -\frac{m+2c\,x}{4x^2\xi}, \quad \beta = -\frac{3m+2c\,x}{4x^2\xi}\,, \quad \psi_2 = \frac{m}{x^3}. \tag{30}$$

a). $c = 0$:

Under a rotation type III with $A = Ln\,x$ and $B = 90^\circ$ we obtain the fulfillment of (2) for:

$$\tilde{\rho} = \tilde{\mu} = \tilde{\gamma} = \tilde{\varepsilon} = 0, \qquad \tilde{\tau} = 4\tilde{\alpha} = i\frac{\xi}{x}\,, \qquad \xi^2 = \frac{m}{x}\,, \qquad \tilde{\psi}_2 = -4\tilde{\pi}\,\tilde{\beta}. \tag{31}$$

b). $c = \pm\frac{1}{2}$:

We employ the relations (8) for $A = Ln\,(\frac{r}{x\sqrt{2}})$ and $B = 90^\circ$ to verify (2) with:

$$\tilde{\rho} = \tilde{\mu} = 0, \ \tilde{\varepsilon} = \frac{1}{2x\sqrt{2}}\,, \ \tilde{\tau} = \frac{i\,\xi}{x}\,, \quad \tilde{\alpha} = \frac{i\,m}{4x^2\xi}\,, \quad \tilde{\psi}_2 = \frac{m}{x^3} = -4\tilde{\pi}\,\tilde{\beta}, \quad q = \mp 1, \quad c = \pm\frac{1}{2}. \tag{32}$$

Now it is complete our analysis of the Kinnersley's metrics, therefore in any type D empty spacetime always is possible to find a Newman-Penrose tetrad with the properties (2), in consecuense are immediate the identities (3) of Czapor-McLenaghan [11, 13, 14].

Received May 18, 2016; Accepted June 04, 2016

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