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Degenerate Hermite-Bernoulli Numbers and Polynomials of the Second Kind

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Abstract. In this paper, we introduce a degenerate Hermite-Bernoulli numbers and polynomials of the second kind and develop some elementary properties. We derive some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions.

Keywords: Hermite polynomials, degenerate Bernoulli polynomials of the second kind, degenerate Hermite-Bernoulli polynomials of the second kind, Summation fomulae, Symmetric identities.

1. Introduction

The 2-variable Kampé de Fériet generalization of the Hermite polynomials [2] and [4] reads

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y=-1$ and x is replaced by $2x$. Based on the definition and generating function above, we can define degenerate Hermite polynomials by means of the generating function

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!} \tag{1.3}$$

where $\lambda \neq 0$. Since $(1+\lambda t)^{\frac{1}{\lambda}} \longrightarrow e^t$ as $\lambda \longrightarrow 0$, it is evident that (1.3) reduces to (1.2). That is $H_n(x,y)$ limiting case of $H_n(x,y;\lambda)$ when $\lambda \longrightarrow 0$.
By equating coefficients of t^n on both the sides of (1.3), the following representation of $H_n(x,y;\lambda)$ is obtained

$$H_n(x,y;\lambda) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{(-\frac{x}{\lambda})_{n-2r}(-\frac{y}{\lambda})_r(-\lambda)^{n-r}}{r!(n-2r)!} \tag{1.4}$$

Since $\lim_{\lambda\longrightarrow 0} H_n(x,y;\lambda) = H_n(x,y)$, (1.1) is a limiting case of (1.4).

In [3], Carlitz, L introduced the degenerate Bernoulli polynomials defined by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_n(x|\lambda)\frac{t^n}{n!} \tag{1.5}$$

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When $x = 0$, $\beta_n(\lambda) = \beta_n^{(\alpha)}(\lambda, 0)$ are called the degenerate Bernoulli numbers.
Note that

$$\lim_{\lambda \longrightarrow 0} \beta_n(x|\lambda) = B_n(x), (n \geq 0).$$

Kim and Seo [7] introduced the degenerate Bernoulli polynomials of the second kind defined by

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_n(x|\lambda)\frac{t^n}{n!} \tag{1.6}$$

When $x = 0$ in (1.6), $B_n(\lambda) = B_n(0|\lambda)$ are called the degenerate Bernoulli numbers of the second kind.
Note that

$$\lim_{\lambda \longrightarrow 0} B_n(x|\lambda) = B_n(x)$$

where $B_n(x)$ are called the Bernoulli polynomials (see [6, 8, 9]).

Pathan and Khan [8] introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x, y)$ defined by

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x, y)\frac{t^n}{n!} \tag{1.7}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x, y)$ introduced by Dattoli et al [4, p.386(1.6)] in the form

$$\left(\frac{t}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x, y)\frac{t^n}{n!} \tag{1.8}$$

The Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n, l)x^l, (n \geq 0) \tag{1.9}$$

and the Stirling number of the second kind is defined by generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n)\frac{t^l}{l!} \tag{1.10}$$

A generalized falling factorial sum $\sigma_k(n; \lambda)$ can be defined by the generating function

$$\sum_{k=0}^{\infty} \sigma_k(n; \lambda)\frac{t^k}{k!} = \frac{(1+\lambda t))^{\frac{(n+1)}{\lambda}} - 1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}, \quad \text{(see [10])} \tag{1.11}$$

where $\lim_{\lambda \longrightarrow 0} \sigma_k(n; \lambda) = S_k(n)$.

In this paper, we consider a degenerate Hermite-Bernoulli numbers and polynomials of the second kind ${}_HB_n(x, y|\lambda)$ and derive some identities and formulae related to Hermite-Bernoulli numbers and polynomials of the second kind.

2. Degenerate Hermite-Bernoulli Polynomials of the Second Kind

Let us assume that $\lambda, t \in \mathbb{C}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$. Then, we consider the degenerate Hermite-Bernoulli polynomials of the second kind which are given by the generating function

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_HB_n(x, y|\lambda)\frac{t^n}{n!} \tag{2.1}$$

so that

$$ {}_HB_n(x,y|\lambda) = \sum_{m=0}^{n} \binom{n}{m} B_m(\lambda) H_{n-m}(x,y|\lambda) \tag{2.2} $$

When $x = y = 0$ in (2.1), $B_n(\lambda) = {}_HB_n(0,0|\lambda)$ are called the degenerate Bernoulli numbers of the second kind.
From (2.1), we note that

$$ \sum_{n=0}^{\infty} \lim_{\lambda\longrightarrow 0} {}_HB_n(x,y|\lambda)\frac{t^n}{n!} = \lim_{\lambda\longrightarrow 0} \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!} \tag{2.3} $$

Thus, we get

$$ \lim_{\lambda\longrightarrow 0} {}_HB_n(x,y|\lambda) = {}_HB_n(x,y), (n \geq 0) \tag{2.4} $$

Theorem 2.1. For $n \geq 0$, we have

$$ {}_HB_n(x,y|\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{(m+1)} \lambda^m {}_H\beta_{n-m}(x,y|\lambda) \tag{2.5} $$

Proof. Form (2.1), we have

$$ \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \left(\frac{\log(1+\lambda t}{\lambda t}\right)\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$

$$ = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m+1}\lambda^m t^m\right)\left(\sum_{n=0}^{\infty} {}_H\beta_n(x,y|\lambda)\frac{t^n}{n!}\right) $$

$$ = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{(m+1)}\lambda^m {}_H\beta_{n-m}(x,y|\lambda)\right)\frac{t^n}{n!} \tag{2.6} $$

Comparing the coefficients of equations (2.1) and (2.6), we get the result (2.5).

Theorem 2.2. For $n \geq 0$, we have

$$ {}_HB_n(x,y|\lambda) = \sum_{m=0}^{n} \binom{n}{m} {}_H\beta_{n-m}(x,y|\lambda)\lambda^m D_m(0) \tag{2.7} $$

Proof. Since

$$ \sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!} = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \tag{2.8} $$

$$ = \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\left(\frac{\log(1+\lambda t}{\lambda t}\right) $$

$$ = \sum_{n=0}^{\infty} {}_H\beta_n(x,y|\lambda)\frac{t^n}{n!}\sum_{m=0}^{\infty} D_m(0)\frac{(\lambda t)^m}{m!} $$

where $D_n(x)$ are Daehee polynomials defined by $\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}$ and $D_n(0) = D_n$ are Daehee numbers (see [5]).

Replacing n by $n-m$ in above equation and comparing the coefficient of t^n, we get the result (2.7).

Theorem 2.3. For $n \geq 0$, we have

$$\sum_{m=0}^{n} {}_HB_m(x,y)\lambda^{n-m}S_1(n,m) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{(m+1)} \lambda^m {}_H\beta_{n-m}(x,y|\lambda) \tag{2.9}$$

Proof. Replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (1.8), we have

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \sum_{n=0}^{\infty} {}_HB_n(x,y)\lambda^{-n}\frac{1}{n!}(\log(1+\lambda t))^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} {}_HB_m(x,y)\lambda^{n-m}S_1(n,m) \right) \frac{t^n}{n!} \tag{2.10}$$

On the other hand

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \left(\frac{\log(1+\lambda t)}{\lambda t}\right)\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\right)$$

$$= \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m+1}\lambda^m t^m\right)\left(\sum_{n=0}^{\infty} {}_H\beta_n(x,y|\lambda)\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!}{(m+1)} \lambda^m {}_H\beta_{n-m}(x,y|\lambda) \right) \frac{t^n}{n!} \tag{2.11}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

Theorem 2.4. For $n \geq 0$, we have

$$\frac{{}_HB_{n+1}(x+1,y|\lambda) - {}_HB_{n+1}(x,y|\lambda)}{n+1} = \sum_{m=0}^{n} \binom{n}{m} \frac{\lambda^m(-1)^m m!}{m+1} H_{n-m}(x,y|\lambda) \tag{2.12}$$

Proof. By using the definition (2.1), we have

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} - \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \log(1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$\sum_{n=0}^{\infty} \left[\frac{{}_HB_{n+1}(x+1,y|\lambda) - {}_HB_{n+1}(x,y|\lambda)}{n+1} \right] \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \frac{\lambda^m(-1)^m m!}{m+1} H_{n-m}(x,y|\lambda) \right) \frac{t^n}{n!}$$

Comparing the coefficients of t^n, we get the result (2.12).

Theorem 2.5. For $n \geq 0$ and $d \in \mathbb{N}$, we have

$$ {}_HB_{n,\lambda}(x,y) = d^{n-1}\sum_{a=0}^{d-1} {}_HB_n\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right) \tag{2.13}$$

Proof. From (2.1) in the form

$$\sum_{n=0}^{\infty} {}_HB_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}(1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{a+x}{\lambda}}$$

$$= \frac{1}{d}\left(\sum_{a=0}^{d-1}\frac{\log(1+\lambda t)^{\frac{d}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\right)(1+\lambda t)^{\frac{d}{\lambda}\frac{a+x}{d}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \sum_{n=0}^{\infty}\left(d^{n-1}\sum_{a=0}^{d-1} {}_HB_n(\frac{a+x}{d}, y|\frac{\lambda}{d})\right)\frac{t^n}{n!}, (d \in \mathbb{N})$$

Comparing the coefficients of t^n, we get the result (2.13).

3. Implicit Formulae Involving Degenerate Hermite-Bernoulli Polynomials of the Second Kind

The result of this section present implicit summation formulae for degenerate Hermite-Bernoulli polynomials of the second kind as follows:

Theorem 3.1. The following implicit summation formula involving degenerate Hermite-Bernoulli polynomials of the second kind ${}_HB_n(x,y|\lambda)$ holds true:

$$ {}_HB_n(x+z, y+u|\lambda) = \sum_{s=0}^{n}\binom{n}{s} {}_HB_{n-s}(x,y|\lambda)H_s(z,u|\lambda) \tag{3.1}$$

Proof. We replace x by x+z and y by y+u in (2.1), use (1.3) and rewrite the generating function as

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+z}{\lambda}}(1+\lambda t^2)^{\frac{y+u}{\lambda}} = \sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!}\sum_{s=0}^{\infty}H_s(z,u|\lambda)\frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} {}_HB_n(x+z, y+u|\lambda)\frac{t^n}{n!} \tag{3.2}$$

Now replacing n by $n-s$ in l.h.s and comparing the coefficients of t^n, we get the result (3.1).

Remark 1. By taking the limit as $\lambda \longrightarrow 0$ in (3.1), we have
Corollary 1.

$$ {}_HB_n(x+z, y+u) = \sum_{s=0}^{n}\binom{n}{s} {}_HB_{n-s}(x,y)H_s(z,u) \tag{3.3}$$

Theorem 3.2. The following implicit summation formula involving degenerate Hermite-Bernoulli polynomials of the second kind ${}_HB_n(x,y|\lambda)$ holds true:

$$ {}_HB_n(x,y|\lambda) = \sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]} B_m(1|\lambda)(-\frac{x}{\lambda})_{n-m-2j}(-\lambda)^{n-m-j}(-\frac{y}{\lambda})_j\frac{n!}{m!j!(n-2j-m)!} \tag{3.4}$$

Proof. Applying the definition (2.1) to the term $\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$ and expanding the function $(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$ at $t=0$ yields

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \left(\sum_{m=0}^{\infty} B_m(1|\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(-\frac{x}{\lambda})_n\frac{(-\lambda t)^n}{n!}\right)\left(\sum_{j=0}^{\infty}(-\frac{y}{\lambda})_j\frac{(-\lambda t^2)^j}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} B_m(1|\lambda)(-\frac{x}{\lambda})_{n-m}(-\lambda)^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}(-\frac{y}{\lambda})_j\frac{(-\lambda t^2)^j}{j!}\right)$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m} B_m(1|\lambda)(-\frac{x}{\lambda})_{n-m-2j}(-\lambda)^{n-m-j}(-\frac{y}{\lambda})_j\right)\frac{t^n}{(n-2j)!j!} \tag{3.5}$$

Equating their coefficients of t^n, we get the result (3.4).

Theorem 3.3. The following implicit summation formula involving degenerate Hermite-Bernoulli polynomials of the second kind ${}_HB_n(x,y|\lambda)$ holds true:

$${}_HB_n(x,y|\lambda) = \sum_{m=0}^{n}\binom{n}{m}(-\frac{z}{\lambda})_{n-m}(-\lambda)^{n-m}\,{}_HB_m(x-z,y|\lambda) \tag{3.6}$$

Proof. By exploiting the generating function (2.1), we can write the equation

$$\sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!} = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x-z}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{z}{\lambda}} \tag{3.7}$$

$$= \left(\sum_{m=0}^{\infty} {}_HB_m(x-z,y|\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(-\frac{z}{\lambda})_n\frac{(-\lambda t)^n}{n!}\right)$$

Replacing n by $n-m$ in above equation and equating their coefficients of t^n leads to formula (3.6).

Theorem 3.4. The following implicit summation formula involving degenerate Hermite-Bernoulli polynomials of the second kind ${}_HB_n(x,y|\lambda)$ holds true:

$${}_HB_n(x+1,y|\lambda) = \sum_{r=0}^{n}\binom{n}{r}(-\frac{1}{\lambda})_r(-\lambda)^r\,{}_HB_{n-r}(x,y|\lambda) \tag{3.8}$$

Proof. By the definition of degenerate Hermite-Bernoulli polynomials of the second kind, we have

$$\sum_{n=0}^{\infty} {}_HB_n(x+1,y|\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!} = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1)$$

$$= \left(\sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}(-\frac{1}{\lambda})_r\frac{(-\lambda t)^r}{r!}\right) + \sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_HB_{n-r}(x,y|\lambda)(-\frac{1}{\lambda})_r(-\lambda)^r \frac{t^n}{(n-r)!r!} + \sum_{n=0}^{\infty} {}_HB_n(x,y|\lambda)\frac{t^n}{n!}$$

Finally, equating the coefficients of the like powers of t^n, we get (3.8).

4. Identities for Degenerate Hermite poly-Bernoulli Polynomials of the Second Kind

In this section, we give general symmetry identities for the degenerate Bernoulli polynomials of the second kind $B_n(x|\lambda)$ and the degenerate Hermite Bernoulli polynomials of the second kind ${}_HB_n(x,y|\lambda)$ by applying the generating function(1.6) and (2.1).

Theorem 4.1. Let $a, b > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_HB_{n-m}(bx, b^2y|\lambda) {}_HB_m(ax, a^2y|\lambda)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_HB_{n-m,\lambda}(ax, a^2y|\lambda) {}_HB_m(bx, b^2y|\lambda) \tag{4.1}$$

Proof. Start with

$$g(t) = \left(\frac{(\log(1+\lambda t)^{\frac{a}{\lambda}})(\log(1+\lambda t)^{\frac{b}{\lambda}})}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)}\right)(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}} \tag{4.2}$$

Then the expression for g(t) is symmetric in a and b and we can expand g(t) into series in two ways to obtain

$$g(t) = \sum_{n=0}^{\infty} {}_HB_n(bx, b^2y|\lambda)\frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_HB_m(ax, a^2y|\lambda)\frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m {}_HB_{n-m}(bx, b^2y|\lambda) {}_HB_m(ax, a^2y|\lambda)\right)\frac{t^n}{n!}$$

On the similar lines we can show that

$$g(t) = \sum_{n=0}^{\infty} {}_HB_n(ax, a^2y|\lambda)\frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_HB_m(bx, b^2y|\lambda)\frac{(at)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_HB_{n-m}(ax, a^2y|\lambda) {}_HB_m(bx, b^2y|\lambda)\right)\frac{t^n}{n!}$$

Comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive the desired result.

Remark 1. By setting $b = 1$ in Theorem 4.1, we immediately following result
Corollary.

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} {}_HB_{n-m}(x, y|\lambda) {}_HB_m(ax, a^2y|\lambda)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m {}_HB_{n-m}(ax, a^2y|\lambda) {}_HB_m(x, y|\lambda) \tag{4.3}$$

Theorem 4.2. For all integers $a > 0, b > 0$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_H B_{n-k}(bx, b^2 z|\lambda) \sum_{i=0}^{k} \binom{k}{i} \sigma_i(a-1|\lambda) B_{k-i}(ay|\lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_H B_{n-k}(ax, a^2 z|\lambda) \sum_{i=0}^{k} \binom{k}{i} \sigma_i(b-1|\lambda) B_{k-i}(by|\lambda) \tag{4.4}$$

where generalized falling factorial sum $\sigma_k(n|\lambda)$ is given by (1.11).
Proof. We now use

$$g(t) = \frac{(\log(1+\lambda t)^{\frac{a}{\lambda}})(\log(1+\lambda t)^{\frac{b}{\lambda}})((1+\lambda t)^{\frac{ab}{\lambda}} - 1)(1+\lambda t)^{\frac{ab(x+y)}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}}{((1+\lambda t)^{\frac{a}{\lambda}} - 1)((1+\lambda t)^{\frac{b}{\lambda}} - 1)^2} \tag{4.5}$$

to find that

$$g(t) = \left(\frac{\log(1+\lambda t)^{\frac{a}{\lambda}}}{(1+\lambda t)^{\frac{a}{\lambda}} - 1}\right)(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}} \left(\frac{(1+\lambda t)^{\frac{ab}{\lambda}} - 1}{(1+\lambda t)^{\frac{b}{\lambda}} - 1}\right)$$

$$\left(\frac{\log(1+\lambda t)^{\frac{b}{\lambda}}}{(1+\lambda t)^{\frac{b}{\lambda}} - 1}\right)(1+\lambda t)^{\frac{aby}{\lambda}} \tag{4.6}$$

$$= \sum_{n=0}^{\infty} {}_H B_n(bx, b^2 z|\lambda) \frac{(at)^n}{n!} \sum_{i=0}^{\infty} \sigma_i(a-1|\lambda) \frac{(bt)^i}{i!} \sum_{k=0}^{\infty} B_k(ay|\lambda) \frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_H B_{n-k}(bx, b^2 z|\lambda) \sum_{i=0}^{k} \binom{k}{i} \sigma_i(a-1|\lambda) B_{k-i}(ay|\lambda) \right) \frac{t^n}{n!} \tag{4.7}$$

Using a similar plan, we get

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_H B_{n-k}(ax, a^2 z|\lambda) \sum_{i=0}^{k} \binom{k}{i} \sigma_i(b-1|\lambda) B_{k-i}(by|\lambda) \right) \frac{t^n}{n!} \tag{4.8}$$

Finally equating the coefficients of t^n on the right hand sides of last two equations, we desired result (4.4).

Theorem 4.3. For all integers $a > 0, b > 0$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sum_{i=0}^{a-1} {}_H B_k(bx + \frac{b}{a} i, b^2 z|\lambda) B_{n-k}(ay|\lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{i=0}^{b-1} {}_H B_k(ax + \frac{a}{b} i, a^2 z|\lambda) B_{n-k}(by|\lambda) \tag{4.9}$$

Proof. From (4.6), g(t) can also be expanded as

$$= \left(\sum_{n=0}^{\infty} \sum_{i=0}^{a-1} {}_H B_n(bx + \frac{b}{a} i, b^2 z|\lambda) \frac{(at)^n}{n!} \right) \left(\sum_{n=0}^{\infty} B_n(ay|\lambda) \frac{(bt)^n}{n!} \right) \tag{4.10}$$

Using a similar plan, we get

$$g(t) = \left(\sum_{n=0}^{\infty} \sum_{i=0}^{b-1} {}_H B_n(ax + \frac{a}{b} i, a^2 z|\lambda) \frac{(bt)^n}{n!} \right) \left(\sum_{n=0}^{\infty} B_n(by|\lambda) \frac{(at)^n}{n!} \right) \tag{4.11}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive the desired result.

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