Article

Aharonov–Bohm Effect under Self-duality Condition of Electromagnetic Field

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Abstract

We discuss the situation of Aharonov–Bohm effect under self-duality condition of electromagnetic field, where there is no exact cancellation of the AB(t) phase shift because this phase shift is complex. This means that the time-dependent AB effect, magnetic AB phase shift is not canceled out by a phase shift coming from the Lorentz force with $E=-\partial_t A$. Our approach can be compared with the results obtained by other authors which have shown that AB(t) effect vanishes for Abelian electromagnetic plane waves.

Keywords: Aharonov–Bohm effect, electromagnetic field, self-duality, phase shift.

1. Introduction

The Aharonov–Bohm effect (AB) [1] has attracted great attention of theorists and experimenters for many years. The interest is caused because, in the theory of gauge fields, the popular belief is that only nontrivial field strength rather than the potentials themselves that are not gauge-invariant can cause the observable effects. However, Aharonov and Bohm demonstrated that the integral of a gauge field along a closed loop can produce the observable effects; the effect produced by the magnetic potential was confirmed experimentally [2-4]. The AB effect is a purely quantum mechanical effect; the original (classified as Type-I) AB-phase shift exists in experimental conditions where the electromagnetic fields and forces are zero. It is the absence of forces that makes the AB effect entirely quantum mechanical. Although the AB-phase shift has been proved unambiguously, the absence of forces in Type-I AB-effects has never been exhibited [1, 2].

Recently, the AB effect with a time-dependent magnetic field has been investigated [5-11], to show that a cancellation of phases occurs in the AB effect with a time-dependent magnetic field. An extra phase coming from the electric field, $E=-\partial_t A$, outside the solenoid cancels out the phase shift of the time-dependent magnetic field, i.e., an exact cancellation of the AB phase shift by means of the phase shift coming from the direct Lorentz force. In this framework, the time-

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dependent AB effect can be considered as another AB(t) effect. Indeed, the normal AB effects are in situations that a charged particle is moving through a region without magnetic and electric fields, while this AB(t) effects are when the charged particle develops an AB phase passing through a region of space with non-zero fields [10]. In [11] the authors have shown that AB(t) effect due to electromagnetic plane waves vanishes under some conditions in terms of the parameters of the system like frequency of the electromagnetic wave, the size of the space–time loop, and amplitude of the electromagnetic wave. In this paper we discuss the situation of Aharonov–Bohm effect under self-duality condition of electromagnetic field, where we have not an exact cancellation of the AB(t) phase shift.

The outline of this paper is as follows. Section 2 presents a description of the self dual field configuration and in Sec. 3 we generalize the Abelian AB effect due to a time-dependent self dual Abelian field configuration. We prove that the AB phase factor no remains equal to zero up to the first order when considering the time-varying vector fields. In Sec. 4 we study the quantization of the magnetic flux in a superconducting ring under self-dual field configuration. Conclusions will be presented in the last Section.

2. Self-dual electromagnetic field

To the best of our knowledge, self-dual configurations of electromagnetic fields have not been considered in classical electrodynamics; it is instructive to consider self-dual fields in this simpler context. Our analysis will show the utility of this concept in problems of experimental interest, for which a stable quasilocalized configuration of the free electromagnetic field is likely to be relevant. We will obtain a field configuration similar to [12-14] but following an alternative route based on the idea of self-duality [15]. There are several reasons for considering self-dual fields in classical electrodynamics: Self-dual solutions are readily calculated and possess trivial energy-momentum properties, and the desired free field configurations are obtainable as superposition of self-dual and anti-self-dual constituents so that the resulting spectral properties may be easily controlled.

To simplify our notation as much as possible, we choose the Gaussian system of units and set the speed of light equal to unity. An electromagnetic field is self-dual/anti-self-dual if:

$$i\mathbf{E} = \pm\mathbf{B}. \tag{1}$$

Is it possible to obtain equation (1)? The answer is yes, if we consider free electromagnetic fields governed by the homogeneous Maxwell equations with the operator $\frac{\partial}{c\partial t}$ transformed to

$\frac{\partial}{c\partial t}(1+T\nabla\times)$ where T is the chiral factor and $1+T\nabla\times=1/2$; under this condition $\mathbf{E}$ is parallel to $\mathbf{B}$:

$$\nabla\times\mathbf{E}=-\frac{\partial}{c\partial t}(1+T\nabla\times)\mathbf{B}, \tag{2}$$

$$\nabla\cdot\mathbf{E}=0, \tag{3}$$

$$\nabla\times\mathbf{B}=\frac{\partial}{c\partial t}(1+T\nabla\times)\mathbf{E}, \tag{4}$$

$$\nabla\cdot\mathbf{B}=0. \tag{5}$$

Let some field configuration be self-dual; if this field obeys (4) and (5), it automatically verifies (2) and (3). Maxwell's equations are linear, hence any superposition of self-dual and anti-self-dual solutions is a further solution. The condition that a field configuration is self-dual is not invariant under the parity transformation $\mathbf{r}\rightarrow\mathbf{-r}$ because of the opposite parity properties of the electric and magnetic field; the mirror-image configuration is anti-self-dual. As will become clear, the physically relevant configurations are represented by a sum of self-dual and anti-self-dual solutions, which is invariant under the parity transformation.

Let us express the electric field intensity $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in terms of scalar and vector potentials V and $\mathbf{A}$. Then the self-duality condition (1) becomes:

$$\pm(\nabla V+\frac{\partial\mathbf{A}}{c\partial t})=-\nabla\times\mathbf{A}\,. \tag{6}$$

If we fix the gauge $V=0$, then (6) reduces to:

$$\pm\frac{\partial\mathbf{A}}{c\partial t}=-\nabla\times\mathbf{A}\,; \tag{7}$$

Because the self-duality condition (7) is a linear first-order partial differential equation, it is simpler to solve than the second-order equations that result from Maxwell's eqs. (2)-(5).

Note that given an antisymmetric field $F_{\mu\nu}$ in Minkowski space, the self-duality condition (1) can be expressed as:

$$*F_{\mu\nu}=\pm iF_{\mu\nu}\,, \tag{8}$$

where the Hodge dual field $*F_{\mu\nu}$ is defined by $*F_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Eq. (8) is identical to (1) because **E** and **B** are expressed in terms of $F_{\mu\nu}$ as $E_i = F_{0i}$ and $B_i = (1/2)\varepsilon_{ijk}F_{jk}$. If the Bianchi identity:

$$\partial_\mu * F^{\mu\nu} = 0, \tag{9}$$

is compared with the equations of motion for a free electromagnetic field:

$$\partial_\mu F^{\mu\nu} = 0, \tag{10}$$

it becomes apparent that if $F_{\mu\nu}$ obeys (9) then $F_{\mu\nu}$ automatically satisfies (10). The imaginary factor *i* in the definition of self-duality (8) is unavoidable if we work in Minkowski space where $**F = -F$.

Therefore, in Minkowski space self-dual field configurations contain complex-valued fields; in contrast, in Euclidean space the factor of *i* is absent and self-dual fields can be real. We note that fundamental physical laws are usually formulated in the form of differential equations with real coefficients. However, this formulation does not necessarily implies that every solution to such equations is real; the only *a priori* constraint, stemming from the fact that the coefficients of the differential equations are real, is that each complex solution is accompanied by a complex-conjugate solution. Complex fields occur only as pairs of complex-conjugate solutions.

3. Time-dependent AB effect for self-dual Abelian gauge fields

The relativistic form of the AB phase factor can be written as follows:

$$\theta = \exp(\frac{e}{\hbar}\oint A_\mu dx^\mu) = \exp(\frac{e}{\hbar}\oint V dt - \mathbf{A}\cdot dx), \tag{11}$$

where A^μ is the Abelian gauge field that might be transformed under the $U(1)$ group via:

$$A^\mu \rightarrow A^\mu + \partial^\mu \zeta \ , \tag{12}$$

such that ζ is a transformation function of space–time coordinates [12, 13]. By using Stokes' theorem, we may rewrite the phase factor in a 2-form structure, stating that the integral of a differential form A'over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative dA' over the whole of Ω, which may be expressed as:

$$\int_{\partial\Omega} A^* = \int_{\Omega} dA^*, \tag{13}$$

where A^* and dA^* are p -form and $(p+1)$-form, respectively. One could also define the 1-form as $A^*=A=A_\mu dx^\mu$ and 2-form $dA^*=dA$ as the Faraday 2-form F by

$$\begin{aligned} dA = {}^*F = (E_x dx + E_y dy + E_z dz) \wedge dt + \\ B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \end{aligned}, \tag{14}$$

where E and B are the electric and magnetic fields, respectively. Therefore, (11) can be rewritten as in (15) below:

$$\theta = \exp(\frac{e}{\hbar}\oint A_\mu dx^\mu) = \exp(-\frac{e}{2\hbar}\int {}^*F_{\mu\nu} dx^\mu \wedge dx^\nu). \tag{15}$$

This expression plays a key role in the study of the AB phase factor when considering the time-dependent Abelian gauge fields. Time-dependent AB effect is based on constructing a subspace in a space–time in which the four-vector potential depends on time; both the electric and the magnetic effects depend on the particle's particular path in this subspace [4]. We assume that the magnetic field inside the solenoid is time-dependent so that the vector potential A will be time-dependent outside the solenoid. However, based on Maxwell's equation, i.e., $E=-\partial_t A$ with $V=0$, an electric field is also created outside the solenoid (we have assumed the scalar potential field V to be zero). Thus from (1), (8), (14) and (15), the magnetic phase factor is obtained by:

$$\frac{e}{\hbar}\int B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy = \frac{e}{\hbar}\int B(x,t)\cdot dS, \tag{16}$$

and the electric part of the phase is given by:

$$\frac{e}{\hbar}\int (E_x dx + E_y dy + E_z dz) \wedge dt = -\frac{e}{\hbar}\int iB(x,t)\cdot dS, \tag{17}$$

where we have replaced the electric field by $-\partial_t A$. It is clear that the AB phase shift for a time-varying magnetic field does not vanish; this means that the magnetic AB phase shift θ is complex and it is not canceled out by a phase shift coming from the Lorentz force associated with the electric field, $E=-\partial_t A$, outside the solenoid, This result is different to the case where **E=B** [5-10] where we have ${}^*F_{\mu\nu} = F_{\mu\nu}$, **E=B**, with c=1 and

$$\frac{e}{\hbar}\int B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy = \frac{e}{\hbar}\int B(x,t)\cdot dS$$

and $\frac{e}{\hbar}\int (E_x dx + E_y dy + E_z dz) \wedge dt = -\frac{e}{\hbar}\int B(x,t)\cdot dS$, so:

$$\theta = \frac{e}{\hbar}\int B(x,t)\cdot dS - \frac{e}{\hbar}\int B(x,t)\cdot dS = 0. \qquad (18)$$

That is, the AB effect with a time-dependent magnetic field obtained in [5-11], shows that a cancellation of phases occurs in the AB effect $\theta = 0$ with a time-dependent magnetic field. An extra phase coming from the electric field, $E = -\partial_t A$, outside the solenoid cancels out the phase shift of the time-dependent magnetic field.

4. Conclusion

We have discussed Aharonov–Bohm effect under self-duality condition of electromagnetic field, $*F_{\mu\nu} = \pm iF_{\mu\nu}$, where we have not an exact cancellation of the AB(t) phase shift because it is complex. This means that the time-dependent AB effect, magnetic AB phase shift is not canceled out by a phase shift coming from the Lorentz force associated with the electric field, $E = -\partial_t A$, $V = 0$. This result can be compared with the obtained by other authors which have shown that AB(t) effect vanishes for Abelian electromagnetic plane waves [5-11], that is, there is an exact cancellation of the magnetic and electric AB phase shifts so that one finds no net phase shift differences coming from the time–dependent electromagnetic field.

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