**Article**

# Twistorial Lift of Kähler Action

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**Abstract**

The critical question for the final formulation of TGD was simple: How to lift the dynamics of Kähler action to twistorial dynamics for twistor bundle having space-time surface as base space represented as 6-D surface in the product of twistor spaces of $M^4$ and $CP_2$? Contrary to expectations, this formulation turned out to be much more than alternative formalism. It led to the identification of cosmological constant and Planck length and more detailed understanding of gravitational interactions in TGD framework. What is most remarkable is that TGD as a theory of everything is completely unique from the condition that the twistor formulation exists. This cannot be said about competitors of TGD.

## 1 Introduction

It took one decade for Einstein to find the final mathematical formulation of General Relativity Theory (GRT). Immediately after having found the final formulation, he predicted gravitional waves, which are easy to discover from the linearized equations. One century later they have been found. Theoretician must be long aged if he wants to enjoy the fruits of his labor.

One should not compare oneself with Gods like Einstein (the nasty colleagues certainly notice this) but since I am totally crazy (ask colleagues) I talk about both us in the same paragraph. In TGD the process of finding final formulation (see `http://tgdtheory.fi/public_html/articles/diagrams.pdf`) [8][5] took almost four decades and now I dare say that I have finally found *the* formulation and the following I try to summarize it. I have done my best to organize the text to a readable form and I apologize if I have not succeeded completely. An entire flood of ideas emerged and they are still developing. This makes documentation difficult.

## 2 Some background

To understand how this is so important I describe briefly the background.

1. Recall that the formulation of classical TGD in terms of Kähler action emerged around 1990 and is therefore quarter century old now. I speak fluently about preferred extremals of Kähler action and I understand reasonably well the dynamics of Kähler action [4]. But about how gravitational constant and cosmological constant emerge from this dynamics I have had only ideas.

2. This dynamics has one very non-standard feature: huge vacuum degeneracy. All 4-surfaces that have $CP_2$ projection, which is so called Lagrangian sub-manifold of $CP_2$ having vanishing induced Kähler form is vacuum extemal. By applying diffeomorphisms of $M^4$ and symplectic transformations of $CP_2$ acting like U(1) gauge transformations one obtains new vacuum extremals. For instance, for the deformations of canonically imbedded empty Minkowski space Kähler action density can be approximated by a fourth order polynomial in $CP_2$ coordinates and their gradients and perturbation theory fails completely since propagator does not exist.

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3. This spoiled completely the hopes about ordinary quantization of the theory and eventually inspired the idea about world of classical worlds (WCW) [7] and led to a beautiful vision about quantum theory as purely classical theory for spinor fields in WCW representing physical states. I could have of course considered the possibility of adding to the action a small volume term to obtain perturbation theory - much like in case of branes - but it looked incredibly ugly. I can only congratulate myself that I refused to consider this possibility although this volume term now emerges from twistorial variant of Kähler action! It had prevented me from discovering WCW and many other deep ideas.

# 3 Only $M^4$ and $CP_2$ allow twistor space with Kähler structure

The situation began to change few years ago as I realized that twistors might be central for understanding of quantum TGD [5].

1. I am not a jedi master in perturbative QFT and formulas hate me. Furthermore, I do not believe on $\mathcal{N} = 4$ SUSY except as a beautiful model able to express some very profound ideas, which have not yet reached our conscious mind. Just the strange beauty of findings of Nima Arkani-Hamed [2, 3] and other pioneers made me convinced that twistors are the key to progress.

2. Year or two ago came the crucial discovery. I learned that $S^4$ (and also conformally compactified version of Minkowski space $M^4$) and $CP_2$ are completely unique 4-D spaces in that only they allow twistor space with Kähler structure [1]. This was discovered by Hitchin. Ironically, this had been discovered roughly year after I ended up with $CP_2$! Somehow I had failed to learn about this. It was immediately clear that this is something incredibly profound and must mean that TGD a twistorially unique.

Precisely how? should have been the immediate question. For some funny reason I did not make quite that question to which the answer would have been totally obvious. I however realized that I should lift space-time surfaces to their 6-D twistor spaces and represent them as 6-D surfaces in the 12-D twistor space of imbedding space inhering their twistor structure from that for the 12-D space. I indeed proposed how to define induction of twistor structure as analog for the induction of metric and spinor structure.

I did not however continue with the obvious question: How to define dynamics for these 6-D twistor spaces identifiable sphere bundles over space-time surfaces?. This question contains its own answer. Twistor structure involves the identification of an antisymmetric tensor defining preferred quaternionic imaginary unit representing it geometrically. The Kähler form of 12-D twistor space projected to space-time surface should define this preferred imaginary unit. TGD exists only for $M^4 \times CP_2$ and is therefore a completely unique theory. The dynamics is determined by 6-D action variant of Kähler action for 6-D surfaces in twistor space of $M^4 \times CP_2$. I want to repeat: TGD is completely unique if one accepts twistorial formulation.

# 4 The existence of twistorial formulation makes TGD unique and leads to unification of gravitation and standard model

After this everything followed within week even though I made all imaginable wrong guesses.

1. 6-D Kähler action has dimension length squared and must be multiplied by a constant with dimensions of 1/length squared. This constant, call it $1/L^2$, is highly analogous to cosmological constant but is coupling constant like parameter since one can replace it with the dimensionless ratio $\epsilon^2 = R^2/L^2$, $R$ radius of $CP_2$. $\epsilon^2$ is analogous to critical temperature (or perhaps better, critical value of inverse pressure) just like Kähler coupling strength and is expected to have spectrum labelled by p-adic primes $p \simeq 2^k$, $k$ prime, just like Kähler coupling strength. The values of $L$

would be most naturally p-adic length scales and cosmological constant would in first approximation decrease with cosmological time $t$ as $1/t^2$.

The incredibly small cosmological constant (not so in the early Universe) would not be just some nasty trick of Universe making theoreticians crazy but needed to remove the vacuum degeneracy of Kähler action in dimensionally reduced dynamics giving rise to 4-volume as additional term in action and making also perturbation theory around canonically imbedded $M^4$ possible. Obviously this action would give rise to the analog of kinetic term for gravitons.

Volume term of course gives just the geometric counterpart of wave equation. In fact, all known extremals of Kähler action are minimal surfaces so that in this sense nothing new has emerged! A coupling between dynamics of volume term and Kähler action is present for more general extremals and Kähler coupling strength and cosmological constant do not completely disappear from the classical dynamics.

2. How the cosmological constant emerges from TGD framework was not at all so trivial as I sloppily thought first. The key point is that the 6-D Kähler action contains two terms.

   (a) The first term is essentially the ordinary Kähler action multiplied by the area of $S^2(X^4)$ which is compensated by the length scale, which can be taken to be the area $4\pi R^2(M^4)$ of $S^2(M^4)$. This makes sense for winding numbers $(w_1, w_2) = (1, 0)$ meaning that $S^2(CP_2)$ is effectively absent but $S^2(M^4)$ is present.

   (b) Second term is the analog of Kähler action assignable assignable to the projection of $S^2(M^4)$ Kähler form. The corresponding Kähler coupling strength $\alpha_K(M^4)$ is huge - so huge that one has

$$\alpha_K(M^4) 4\pi R^2(M^4) \equiv L^2 \quad , \tag{4.1}$$

   where $1/L^2$ is of the order of cosmological constant and thus of the order of the size of the recent Universe. $\alpha_K(M^4)$ is also analogous to critical temperature and the earlier hypothesis that the values of $L$ correspond to p-adic length scales implies that the values of come as $\alpha_K(M^4) \propto p \simeq 2^k$, $p$ prime, $k$ prime.

   (c) The Kähler form assignable to $M^4$ is not assumed to contribute to the action since it does not contribute to spinor connection of $M^4$. One can of course ask whether it could be present. For canonically imbedded $M^4$ self-duality implies that this contribution vanishes and for vacuum extremals of ordinary Kähler action this contribution is small.Breaking of Lorentz invariance is however a possible problem. If $\alpha_K(M^4)$ is given by above expression, then this contribution is extremely small.

   Hence one can consider the possibility that the action is just the sum of full 6-D Kähler actions assignable to $T(M^4)$ and $T(CP_2)$ but with different values of $\alpha_K$ if one has $(w_1, w_2) = (n, 0)$. Also other $w_2 \neq 0$ is possible but corresponds to gigantic cosmological constant.

3. Also other fundamental lengths pop up: the radii on $S^2(M^4)$ and $S^2(CP_2)$. The radius of $S^2(CP_2)$ is essentially $CP_2$ radius from the definition of twistor space but what about $S^2(M^4)$? Here I had to think thoroughly what the twistor space of $M^4$ is. It turned out that the radius is most naturally what Planck length mystic would guess it to be: essentially Planck length $l_P$. Planck length would emerge from the theory as a purely classical scale! Only two weeks ago I explained that Planck length is purely quantal emergent length scale! However, Planck mass and Newton's constant would emerge as quantal parameters since they depend on Planck constant when expressed in terms of Planck length, $\hbar$ and c.

Twistorialization thus brings all the basic scales of gravitation. Earlier I had p-adic length scales emerging from the successful p-adic mass calculations explaining the mass spectra of particles and the $CP_2$ inspired vision about how gauge coupling strengths and their evolution emerge from quantum criticality. My original belief that $G$ and $\Lambda$ would emerge from the dynamics of Kähler action alone was therefore wrong.

4. I had also to learn also what twistor space $T(M^4)$ really is! For $CP_2$ and $S^2$ there are no problems but how to go to the Minkowskian signature - this was the problem. Also here I did all wrong trials. The solution of the problem was simple.

   I had actually found the solution for more than one and half decades ago while studying so called massless extremals (MEs) representing radiation in TGD Universe. Also the study of so called $M^8 - H$ duality and the notion of quaternionic structure had led to what I call Hamilton-Jacobi (H-J) structure [6] generalizing Euclidian 4-space $E^4$ with complex structure to its Minkowskian variant.

   One must first construct $M^4$ with H-J structure and then lift it to twistor space. H-J structure in Minkowski space means the existence of a distribution of spatially varying decompositions $M^4 = M^2(x) \oplus E^2(x)$ of the tangent space of $X^4$ to direct orthogonal sum of local 2-D Minkowski space $M^2(x)$ and and orthogonal Euclidian 2-space $E^2(x)$. This distribution must be integrable meaning that $M^2(x)$ and $E^2(x)$ serve as tangent spaces for 2-D surfaces. Euclidian 2-space allows complex structure and complex coordinates $(z, \overline{z})$. $M^2$ allows hyper-complex structure and hyper-complex coordinates, which are nothing but light-like coordinates $(u = t - z, v = t + z)$ such that metric of $M^2$ is of from $ds^2 = 2dudv$.

   The construction of twistor space looks now rather trivial. Any antisymmetric tensor in the space $E^3$ orthogonal to time like $t$ defines direction that it is point of the sphere defining the twistor space fiber. Metric is induced from the metric for this kind of tensors defined by $M^4$ metric. The covariantly constant two-form of $E^2$ defines preferred quaternionic imaginary unit. This is also familiar from number theoretic vision demanding its existence. The vision about preferred extremals of Kähler action as quaternionic 4-surfaces of octonionic 8-space relies on this vision. In particular, the twistorial sphere is sphere -not hyperbolic sphere with signature (1,-1) as I believed for 24 hours - and it has metric signature (-1,-1) rather than being time-like!

5. I had also to learn what the induction of twistor structure means concretely. The preferred quaternionic imaginary unit should be represented as a projection of Kähler form of 12-D twistor space $T(H)$. The preferred imaginary unit defining twistor structure as sum of projections of both $T(CP_2)$ and $T(M^4)$ Kähler forms would guarantee that vacuum extremals like canonically imbedded $M^4$ for which $T(CP_2)$ Kähler form contributes nothing have well-defined twistor structure. $T(M^4)$ or $T(CP_2)$ are treated completely symmetrically.

6. For Kähler action $M^4 - CP_2$ symmetry does not make sense. 4-D Kähler action to which 6-D Kähler action dimensionally reduces can depend on $CP_2$ Kähler form only. I have also considered the possibility of covariantly constant self-dual $M^4$ term in Kähler action but given it up because of problems with Lorentz invariance. One should couple the gauge potential of $M^4$ Kähler form to induced spinors. This would mean the existence of vacuum gauge fields coupling to sigma matrices of $M^4$ so that the gauge grop would be non-compact $SO(3,1)$ leading to a breakdown of unitarity.

   Hence it seems clear that only the projection of $T(CP_2)$ part of Kähler form of $T(H)$ can appear in 6-D Kähler action. This option breaks the symmetry between $M^4$ and $CP_2$ at the level of dynamics but is physically unavoidable and is also mathematically completely acceptable. I cannot but accept the situation.

7. An important point to notice is that the radius of the sphere associated with the twistor space of $X^4$ is dynamical and cosmological considerations suggest that this radius increases during cosmic

evolution from Planck length to $CP_2$ scale. Also the homotopy class of the map of this sphere to the product of spheres $S^2$ associated with $T(M^4)$ and $R(CP_2)$ is fundamental.

# 5 A connection with the hierarchy of Planck constants?

A connection with the hierarchy of Planck constants is highly suggestive. Since also a connection with the p-adic length scale hierarchy suggests itself for the hierarchy of p-adic length scales it seems that both length scale hierarchies might find first principle explanation in terms of twistorial lift of Kähler action.

1. Cosmological considerations encourage to think that $R_1 \simeq l_P$ and $R_2 \simeq R$ hold true. One would have in early cosmology $(w_1, w_2) = (1, 0)$ and later $(w_1, w_2) = (0, 1)$ guaranteeing $R_D$ grows from $l_P$ to $R$ during cosmological evolution. These situations would correspond the solutions $(w_1 = n, 0)$ and $(0, w_2 = n)$ one has $A = n4\pi R_1^2$ and $A = n \times 4\pi R_2^2$ and both Kähler coupling strengths are scaled down to $\alpha_K/n$. For $\hbar_{eff}/h = n$ exactly the same thing happens!

   There are further intriguing similarities. $h_{eff}/h = n$ is assumed to correspond *multi-sheeted* (to be distinguished from *many-sheeted*!) covering space structure for space-time surface. Now one has covering space defined by the lift $S^2(X^4) \rightarrow S^2(M4) \times S^2(CP_2)$. These lifts define also lifts of space-time surfaces.

   Could the hierarchy of Planck constants correspond to the twistorial surfaces for which $S^2(M^4)$ and $S^(CP_2)$ are identified in 1-1 manner? The assumption has been that the $n$-fold multi-sheeted coverings of space-time surface for $h_{eff}/h = n$ are singular at the ends of space-time surfaces at upper and lower boundaries if causal diamond (CD). Could one consider more precise definition of twistor space in such a manner that $CD$ replaces $M^4$ and the covering becomes singular at the light-like boundaries of CD - the branches of space-time surface would collapse to single one. What could this collapse mean geometrically? Or should one give up the assumption about singular nature of the covering used to distinguishes many-sheetedness from multi-sheetedness.

2. $w_1 = w_2 = w$ is essentially the first proposal for conditions associated with the lifting of twistor space structure. $w_1 = w_2 = n$ gives $ds^2 = (R_1^2 + R_2^2)(d\theta^2 + w^2 d\phi^2)$ and $A = n \times 4\pi(R_1^2 + R_2^2)$. Also now Kähler coupling strength is scaled down to $\alpha/n$. Again a connection with the hierarchy of Planck constants suggests itself.

3. One can consider also the option $R_1 = R_2$ option giving $ds^2 = R_1^2(2d\theta^2 + (w_1^2 + w_2^2)d\phi^2$. If the integers $w_i$ define Pythagorean square one has $w_1^2 + w_2^2 = n^2$ and one has $R_1 = R_2$ option that one has $A = n \times 4\pi R^2$. Also now the connection with the hierarchy of Planck constants might make sense.

# 6 Twistorial variant for the imbedding space spinor structure

The induction of the spinor structure of imbedding space is in key role in quantum TGD. The question arises whether one should lift also spinor structure to the level of twistor space. If so one must understand how spinors for $T(M^4)$ and $T(CP_2)$ are defined and how the induced spinor structure is induced.

1. In the case of $CP_2$ the definition of spinor structure is rather delicate and one must add to the ordinary spinor connection U(1) part, which corresponds physically to the addition of classical U(1) gauge potential and indeed produces correct electroweak couplings to quarks and leptons. It is assumed that the situation does not change in any essential manner: that is the projections of gauge potentials of spinor connection to the space-time surface give those induced from $M^4 \times CP_2$ spinor connection plus possible other parts coming as a projection from the fiber $S^2(M^2) \times S^2(CP_2)$. As a matter of fact, these other parts should vanish if dimensional reduction is what it is meant to be.

2. The key question is whether the complications due to the fact that the geometries of twistor spaces $T(M^4)$ and $T(CP_2)$ are not quite Cartesian products (in the sense that metric could be reduced to a direct sum of metrics for the base and fiber) can be neglected so that one can treat the sphere bundles approximately as Cartesian products $M^4 \times S^2$ and $CP_2 \times S^2$. This will be assumed in the following but should be carefully proven.

3. Locally the spinors of the twistorspace $T(H)$ are tensor products of imbedding spinors and those for of $S^2(M^4) \times S^2(CP_2)$ expressible also as tensor products of spinors for $S^2(M^4)$ and $S^2(CP_2)$. Obviously, the number of spinor components increases by factor $2 \times 2 = 4$ unless one poses some additional conditions taking care that one has dimensional reduction without the emergence of any new spin like degrees of freedom for which there is no physical evidence. The only possible manner to achieve this is to pose covariant constancy conditions already at the level of twistor spaces $T(M^4)$ and $T(CP_2)$ leaving only single spin state in these degrees of freedom.

4. In $CP_2$ covariant constancy is possible for right-handed neutrino so that $CP_2$ spinor structure can be taken as a model. In the case of $CP_2$ spinors covariant constancy is possible for right-handed neutrino and is essentially due to the presence of U(1) part in spinor connection forced by the fact that the spinor structure does not exist otherwise. Ordinary $S^2$ spinor connection defined by vielbein exists always. One can however add a coupling to a suitable multiple of Kähler potential satisfying the quantization of magnetic charge (the magnetic flux defined by $U(1)$ connection is multiple of $2\pi$ so that its imaginary exponential is unity).

   $S^2$ spinor connections must must have besides ordinary vielbein part determined by $S^2$ metric also $U(1)$ part defined by Kähler form coupled with correct coupling so that the curvature form annihilates the second spin state for both $S^2(M^4)$ and $S^2(CP_2)$. $U(1)$ part of the spinor curvature is proportional to Kähler form $J \propto sin(theta) d\theta d\phi$ so that this is possible. The vielbein and $U(1)$ parts of the spinor curvature ear proportional Pauli spin matrix $\sigma_z = (1, 0; 0, -1)/2$ and unit matrix $(1, 0; 0, 1)$ respectively so that the covariant constancy is possible to satisfy and fixes the spin state uniquely.

5. The covariant derivative for the induced spinors is defined by the sum of projections of spinor gauge potentials for $T(M^4)$ and $T(CP_2)$. With above assumptions the contributions gauge potentials from $T(M^4)$ and $T(CP_2)$ separately annihilate single spinor component. As a consequence there are no constraints on the winding numbers $w_i$, $i = 1, 2$ of the maps $S^2(X^4) \to S^2(M^4)$ and $S^2(X^4) \to S^2(CP_2)$. Winding number $w_i$ corresponds to the imbedding map $(\Theta_i = \theta, \Phi_i = w_i\phi)$.

6. If the square of the Kähler form in fiber degrees of freedom gives metric to that its square is metric, one obtains just the area of $S^2$ from the fiber part of action. This is given by the area $A = 4\pi\sqrt{2(w_1^2 R_1^2 + w_2^2 R_2^2)}$ since the induced metric is given by $ds^2 = (R_1^2 + R_2^2)d\theta^2 + (w_1^2 R_1^2 + w_2^2 R_2^2)d\phi^2$ for $(\Theta_1 = \theta, \Phi = n_1\phi, \Phi_2 = n_2\phi)$.

To sum up, I strongly feel the final formulation of TGD has now emerged and it is now clear that TGD is indeed a quantum theory of gravitation allowing to understand standard model symmetries. The existence of twistorial formulation makes possible gravitation and predicts standard model symmetries. This theory is completely unique from extremely general assumptions. This cannot be said about any competitor of TGD.

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