**Article**

# On the Recursive Formulation of Tau Method Proposed by Issa-Adeniyi

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## Abstract

We show that the usual formulation of the tau method is well adapted to the problems proposed by Issa-Adeniyi, hence it is not necessary to introduce a complicate reformulation of this important Lanczos technique to construct polynomial solutions of ordinary differential equations.

**Keywords**: Tau method, Lanczos-Ortiz, canonical polynomials.

## 1. Introduction

Issa-Adeniyi [1] introduce a reformulation of the tau method [2] to obtain numerical solutions of certain class of problems in ordinary differential equations, for example, to solve:

$$y' - x^2\, y = 0, \qquad y(0) = 1, \tag{1}$$

and with their procedure they construct the following polynomial solution of 5th order:

$$y(x) = 1 + \frac{1}{26\,442\,910\,625}(42\,887\,122\,x - 1\,012\,427\,712\,x^2 + 13\,794\,322\,304\,x^3 - $$

$$-9\,264\,926\,976\,x^4 + 6\,901\,011\,968\,x^5)\,. \tag{2}$$

Here we show that with the usual version of the tau method is possible to study (1) and to give an alternative polynomial solution of fifth order simpler than (2).

## 2. The tau method

The Lanczos algorithm can be successfully applied to linear differential equations of arbitrary order, with the only condition that their coefficients have to be polynomials, with certain boundary conditions, in [-1,1] (strictly speaking this is not a restriction, since a change of scale can always be made) due to the fact that the Chebyshev polynomials $T_j(x)$ [3, 4] vary uniformly

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in the whole interval; if we make the analysis in [0,1] then the tau method remains unaltered, because it is only necessary to employ the modified Chebyshev polynomials $T_k^*(x)$ [5, 6] instead of the $T_k$.

Let us consider the problem:

$$Dy(x) = 0, \tag{3}$$

such that $D$ is a linear differential operator of order $\alpha$, with the initial conditions:

$$y^{(k)}(0), \qquad k = 0, 1, \dots, \alpha - 1. \tag{4}$$

In the next step the Lanczos-Ortiz canonical polynomials [2, 5, 7-9] $Q_m$ are constructed, with the important clarification that $m$ does not necessarily refer to the polynomial order:

$$DQ_m(x) = x^m + R_m(x), \qquad m = 0, 1, 2, \dots \tag{5}$$

where the $R_m$ are known as residual polynomials. It is important to note that there can exist certain values $m_1, m_2, \dots, m_s$ for which the prescription (5) does not work, namely, for which it is not possible to construct the pair $Q_{m_j}$ and $R_{m_j}$ verifying (5). In these cases, the $Q_{m_j}$ are known as indefinite polynomials, and it is convenient to introduce an ensemble $S$ that contains such pathological values:

$$S = \{m_1, m_2, \dots, m_s\}, \tag{6}$$

on the other hand, all the residual polynomials are linear combinations of the different powers $x^{m_j}$, $j = 1, \dots, s$:

$$R_k(x) = C_k^{m_1} x^{m_1} + C_k^{m_2} x^{m_2} + \cdots + C_k^{m_s} x^{m_s}, \quad k = 0,1, \dots, \quad k \notin S, \tag{7}$$

note that in (7) we have $k \neq m_j$ because the $R_{m_j}$ are indefinite.

Here it is accepted that the original problem (3) does not have exact polynomials solutions, which excludes the existence of multiple canonical polynomials. For instance, if two different polynomials $Q_a$ and $Q_b$ provide the same power of $x$, $DQ_a = DQ_b = x^p$, then $D(Q_a - Q_b) = 0$ and $Q_a - Q_b$ will be a polynomial solution of (3). It is possible to give the corresponding extension of the method in the case in which (3) allows exact polynomial solutions. Subsequently, Lanczos proposes to replace the zero in (3) for a small perturbation:

$$D\tilde{y}(x) = H_n(x), \tag{8}$$

where $H_n$ is a $n$-degree polynomial, and $\tilde{y}(x)$ is an exact polynomial solution subjected to the same boundary conditions (4):

$$\tilde{y}^{(k)}(0) = y^{(k)}(0), \qquad k = 0, \dots, \alpha - 1, \tag{9}$$

which in turn, is a good polynomial approximation for the problem (3), with an error uniformly distributed in [-1,1], we can achieve this last property if $H_n$ is written in terms of Chebyshev´s $T_k$:

$$H_n(x) = (\tau_0 + \tau_1 x + \dots + \tau_r x^r)\, T_{n-r}(x), \qquad r = \alpha + s - 1, \tag{10}$$

notice the presence of the ($r$+1) parameters $\tau_j$ that the algorithm itself allows to determine, and whose magnitudes are small because $H_n$ should not deviate much from the zero of the right hand side of (3). Therefore, the quantity of parameters $\tau_j$ depends on the order $\alpha$ of the differential operator $D$, and also of the cardinality of $S$, i.e. of the number of the indefinite canonical polynomials. On the other hand, the Chebyshev polynomial that appears in (10) can be written in the following form:

$$T_{n-r}(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_{n-r} x^{n-r}, \tag{11}$$

its corresponding coefficients $c_k$ are data that take part in different equations of the tau method.

Then, we can express the exact polynomial solution of (8) as:

$$\tilde{y}(x) = \sum_{\substack{m=0 \\ (m+i) \notin S}}^{n-r} c_m \sum_{i=0}^{r} \tau_i\, Q_{m+i}(x), \tag{12}$$

note that ($m$ + $i$) does not belong to (6) $\therefore\ (m+i) \neq m_j,\ j = 1, \dots, s$. If we impose in (12) the boundary conditions (9), we obtain the constraints:

$$\sum_{\substack{m=0 \\ (m+i) \neq m_j}}^{n-r} c_m \sum_{i=0}^{r} \tau_i\, Q_{m+i}^{(k)}(0) = y^{(k)}(0), \qquad k = 0, 1, \dots, \alpha - 1, \tag{13}$$

and the substitution of (12) into (8) provides the relations:

$$\sum_{i=0}^{r} \tau_i \left[ \sum_{\substack{k=0 \\ (k+i) \in S}}^{n-r} c_k\, x^{k+i} - \sum_{\substack{m=0 \\ (m+i) \notin S}}^{n-r} c_m\, R_{m+i}(x) \right] = 0. \tag{14}$$

In (13) there are $\alpha$ conditions, meanwhile (14) implies $s$ constraints (because we should equal the coefficients of the different powers $x^q$ to zero, whose $q$-values are all contained in $S$), making a total of $\alpha$ + $s$ = $r$ + 1 algebraic equations to compute the ($r$ + 1) parameters $\tau_j$, being therefore

(12) completely determined. It should be emphasized that $n$ is a datum, because it can be decided (depending on the problem under analysis) the order of the perturbation, and therefore the tau process provides an exact solution for (8) and (9). Note that the order of $\tilde{y}(x)$ is not necessarily equal to $n$, in fact, this depends on the structure of the differential operator $D$.

If this process of Lanczos is applied to (1), that is, to (8) with $n = 7$, we obtain the quantities:

$$\alpha = 1, \quad m_1 = 0, \; m_2 = 1, \; s = r = 2, \; H_7 = (\tau_0 + \tau_1 x + \tau_2 x^2)\, T_5, \; Q_0 \text{ and } Q_1 \text{ are indefinite},$$

$$T_5(x) = 5x - 20x^3 + 16x^5, \; c_0 = c_2 = c_4 = 0, \; c_1 = 5, \; c_3 = -20, \; c_5 = 16, \; Q_2 = -1,$$

$$Q_3 = -x, \; Q_4 = -x^2, \; Q_5 = -x^3 - 3, \; Q_6 = -x^4 - 4x, \; Q_7 = -x^5 - 5x^2, \; Q_8 = -x^6 - 6x^3 - 18,$$

$$Q_9 = -x^7 - 7x^4 - 28x, \; R_2 = R_5 = R_8 = 0, \; 28R_3 = 7R_6 = R_9 = -28, \; 5R_4 = R_7 = -10x, \quad (15)$$

$$\tau_0 = -\frac{696}{32\,563}, \qquad \tau_1 = -\frac{215}{32\,563}, \qquad \tau_2 = -\frac{32}{32\,563},$$

with the exact solution of (8):

$$\tilde{y} = 1 + \frac{1}{32\,563}(-1\,740\, x^2 + 10\,496\, x^3 + 3\,440\, x^4 + 512\, x^5), \qquad (16)$$

which is an approximate solution of (1), and simpler than (2).

A similar process can be applied to the several problems proposed by Issa-Adeniyi [1], without to modify the usual tau method.

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## References

1. K. Issa, R. B. Adeniyi, *A generalized scheme for the numerical solution of initial value problems in ordinary differential equations by the recursive formulation of tau method*, Int. J. Pure Appl. Maths. **88**, No. 1 (2013) 1-13
2. C. Lanczos, *Trigonometric interpolation of empirical and analytical functions*, J. Math. Phys. **17** (1938) 123-199
3. P. L. Chebyshev, *Théorie des mécanismes connus sous le nom de parallélogrammes*, Mém. Acad. Sci. Pétersb. **7** (1854) 539-568
4. M. A. Snyder, *Chebyshev methods in numerical approximation*, Prentice-Hall, New Jersey (1966)
5. C. Lanczos, *Applied analysis*, Dover, New York (1988) Chap. 7

6. S. B. Barragán, J. López-Bonilla, B. Man Tuladhar, *Powers $x^n$ in terms of modified Chebyshev polynomials*, Kathmandu Univ. J. Sci. Eng. Tech. **6**, No. 2 (2010) 111-114
7. E. L. Ortiz, *On the generation of the canonical polynomials associated with certain linear differential operators*, Res. Rep., Imperial College, London (1964) 1-22
8. E. L. Ortiz, *Canonical polynomials in the Lanczos tau method*, in 'Studies in numerical analysis', Ed. B. K. P. Scaife, Academic Press, New York (1974) 73-93
9. J. P. Boyd, *Chebyshev and Fourier spectral methods*, Dover, New York (2001)