On the Linear Differential Equation of Second Order

Gyan B. Thapa\textsuperscript{1}, A. Domínguez-Pacheco\textsuperscript{2} & J. López-Bonilla\textsuperscript{2*}

\textsuperscript{1}Central Campus, Institute of Engineering, Tribhuvan University, Kathmandu, Nepal
\textsuperscript{2}ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Col. Lindavista 07738, México DF

Abstract

We consider the equation $p(x)y'' + q(x)y' + r(x)y = \phi(x)$ for $p'' - q' + r = 0$, where it is possible to obtain the solutions $y_1$ and $y_2$ of the corresponding homogeneous equation and a particular solution $y_p$ for the original equation, and also for $p'' - q' + r \neq 0$, where we must know $y_1$ to construct $y_2$ and $y_p$ via two integrations of certain differential relation.

Keywords: 2th order, linear differential equation, variation of parameters.

1. Introduction

Here we study the general solution of second order linear differential equation:

$$p(x)y'' + q(x)y' + r(x)y = \phi(x), \quad (1)$$

via an alternative (but equivalent) method to the variation of parameters technique of Newton (Principia)-Bernoulli-Euler-Lagrange [1]. It is convenient to consider two cases:

a). $p'' - q' + r = 0$.

In Sec. 2 we exhibit that the differential expression [2]:

$$\frac{d}{dx} \left[ p^2 W \left( \frac{y}{pw} \right) \right] = \phi, \quad W(x) = \exp \left( - \int^x \frac{q(\xi)}{p(\xi)} d\xi \right), \quad (2)$$

gives the complete solution of (1).

b). $p'' - q' + r \neq 0$.

The Sec. 3 shows that two integrations of: 

\[ \text{Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, México DF} \]
\[ \text{E-mail: jlopezb@ipn.mx} \]
allows to construct the general solution of (1).

2. Case $p'' - q' + r = 0$

In this situation first we calculate the wronskian $W$, and after two successive integrations of (2) we obtain the complete solution of (1):

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

where

$$y_1 = Wp, \quad y_2 = y_1 \int^x \frac{d\xi}{Wp^2} = y_1 \int^x \frac{W}{y_1^2} d\xi, \quad p(x)y_2'' + q(x)y_2' + r(x)y_2 = 0,$$

$$y_p = y_1 \int^x \frac{d\xi}{Wp^2} \int^x \phi(\eta) d\eta = y_2 \int^x \phi(\xi) d\xi - y_1 \int^x \frac{y_\phi}{y_1} d\xi,$$

in harmony with the variation of parameters method [1, 3].

3. Case $p'' - q' + r \neq 0$

Here we need one solution of the homogeneous equation associated to (1), then two integrations of (3) give the general solution (4) such that:

$$y_2(x) = y_1(x) \int^x \frac{W(\xi)}{[y_1(\xi)]^2} d\xi, \quad y_p(x) = y_2(x) \int^x \frac{y_1(\xi)\phi(\xi)}{p(\xi)W(\xi)} d\xi - y_1(x) \int^x \frac{y_2(\xi)\phi(\xi)}{p(\xi)W(\xi)} d\xi,$$

and (6) implies (5) when $y_1 = Wp$. The integration of (3) justifies the traditional ansatz [1, 3] employed in the variation of parameters technique. It is easy to apply our approach to differential equations of third and fourth order [4].

The fundamental differential relation (3) can be deduced via the self-adjoint and exact operators concepts [3, 5, 6] applied to (1) (thus it is not necessary the Lagrange’s ansatz), with the important participation of the expression (2) of Abel-Liouville-Ostrogradski [7] for the wronskian $W = y_1y_2' - y_2y_1'$. 
References