Report

Empty Type D Metrics & Their Lanczos Potential

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Abstract

We employ the Newman-Penrose formalism to construct the Lanczos spin tensor for any type D vacuum spacetime.

Keywords: Lanczos potential, Kinnersley solutions, spin coefficients.

1. Introduction

We shall use the notation and conventions of [1-3]. Lanczos [4, 5] showed that the conformal tensor, in an arbitrary spacetime, is generated by the potential \(K_{\mu\nu\alpha}\) via the following expression for the complex Weyl tensor [2]:

\[
S_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + i *C_{\mu\nu\alpha\beta} = S_{\mu\nu\alpha\beta} - S_{\mu\alpha\nu\beta} + S_{\alpha\beta\mu\nu} - S_{\alpha\beta\nu\mu} + \frac{1}{2} \left[ (H_{\mu\beta} + H_{\beta\mu}) g_{\nu\alpha} - (H_{\mu\alpha} + H_{\alpha\mu}) g_{\nu\beta} + (H_{\nu\alpha} + H_{\alpha\nu}) g_{\mu\beta} - (H_{\nu\beta} + H_{\beta\nu}) g_{\mu\alpha} \right],
\]

with the complex spin tensor \(S_{\mu\nu\alpha} \equiv K_{\mu\nu\alpha} + i *K_{\mu\nu\alpha}\) and \(H_{\mu\nu} \equiv S_{\mu\nu}^\alpha \alpha\).

The Lanczos potential satisfies the algebraic conditions [1]:

\[
K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \quad K_{\mu}^{\nu} = 0, \quad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0,
\]

hence it has 16 independent real components coded into the complex quantities \(\Omega_{r}, \ r = 0, ...,7\), that is, the projections of \(S_{\mu\nu\alpha}\) [1] onto the null tetrad \((l_\mu, n_\mu, m_\mu, \bar{m}_\mu)\) of Newman-Penrose (NP) [1-3, 5-7]:

\[
S_{\mu\nu\alpha} = 2 \left[ \Omega_0 U_{\mu\nu} n_\alpha + \Omega_1 (M_{\mu\nu} n_\alpha - U_{\mu\nu} m_\alpha) + \Omega_2 (V_{\mu\nu} n_\alpha - M_{\mu\nu} m_\alpha) - \Omega_3 V_{\mu\nu} m_\alpha \right].
\]

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\[-\Omega_4 U_{\mu \nu} \bar{m}_\alpha + \Omega_5 (U_{\mu \nu} l_\alpha - M_{\mu \nu} \bar{m}_\alpha) + \Omega_6 (M_{\mu \nu} l_\alpha - V_{\mu \nu} \bar{m}_\alpha) + \Omega_7 V_{\mu \nu} l_\alpha,\]

where \( V_{\mu \nu} = l_\mu \times n_\nu \), \( U_{\mu \nu} = \bar{m}_\mu \times n_\nu \), \( M_{\mu \nu} = n_\mu \times l_\nu + m_\mu \times \bar{m}_\nu \) and:

\[
\begin{align*}
2 \Omega_0 &= S_{(1)(3)(1)}, & 2 \Omega_1 &= S_{(1)(3)(4)}, & 2 \Omega_2 &= S_{(4)(2)(1)}, & 2 \Omega_3 &= S_{(4)(2)(4)}, \\
2 \Omega_4 &= S_{(1)(3)(3)}, & 2 \Omega_5 &= S_{(1)(3)(2)}, & 2 \Omega_6 &= S_{(4)(2)(3)}, & 2 \Omega_7 &= S_{(4)(2)(2)},
\end{align*}
\]

then \( K_{\mu \nu \alpha} = \frac{1}{2} (S_{\mu \nu \alpha} + \bar{S}_{\mu \nu \alpha}) \) is determined if we know the \( \Omega_r \) for a given NP tetrad.

In [8] were obtained the NP components (4) for arbitrary spacetimes of Petrov types 0, N and III, with respect to any canonical tetrad [7, 9]. It is interesting to note that in these cases the \( \Omega_r \) are proportional to the corresponding spin coefficients [3, 6, 7].

Here we deduce the Lanczos potential for the 11 metrics of Kinnersley [10] associated to an arbitrary type D vacuum spacetimes: We work with an adequate canonical null tetrad to achieve certain relationships between its spin coefficients, which allows find a general expression for the \( \Omega_r \).

### 2. Kinnersley’s metrics

We select to \( l^\mu \) and \( n^\mu \) as the two Debever-Penrose principal directions [7, 9] associated with any empty type D solution, then the Goldberg-Sachs theorem [7, 11] implies:

\[
\kappa = \sigma = \lambda = \nu = 0; \tag{5}
\]

besides it is possible to show that, for each metric of Kinnersley [10], exists a scale-rotation (type III [3, 7]) onto this null tetrad such that:

\[
\tau = \pi, \quad \alpha = \beta, \quad \gamma = q \varepsilon, \quad \mu = q \varphi, \quad \psi_2 = 4(\gamma \rho - \pi \beta), \quad q = \pm 1. \tag{6}
\]

If we employ (5), (6), \( \psi_r = 0, \ r \neq 2 \) and the Newman-Penrose equations in the Weyl-Lanczos relations [2]:

\[
\psi_0 = 2[\delta \Omega_0 - D \Omega_4 + (-\bar{\alpha} - 3 \beta + \bar{\pi}) \Omega_0 + 3 \sigma \Omega_1 + (\bar{\rho} + 3 \varepsilon - \bar{\varepsilon}) \Omega_4 - 3 \kappa \Omega_5],
\]

\[
2 \psi_1 = \Delta \Omega_0 + 3 \delta \Omega_1 - \bar{\delta} \Omega_4 - 3 D \Omega_5 - (3 \gamma + \bar{\gamma} + 3 \mu - \bar{\mu}) \Omega_0 + 3(-\bar{\alpha} - \beta + \bar{\pi} + \tau) \Omega_1 +
\]
\[ + 6 \sigma \Omega_2 + (3 \alpha - \bar{\beta} + 3 \pi + \bar{\tau}) \Omega_4 + 3 (\varepsilon - \bar{\varepsilon} + \rho - \bar{\rho}) \Omega_5 - 6 \kappa \Omega_6, \]

\[ \psi_2 = \Delta \Omega_1 + \delta \Omega_2 \delta \Omega_5 - D \Omega_6 - \nu \Omega_0 - (2 \mu - \bar{\mu} + \gamma + \bar{\gamma}) \Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi} + 2 \tau) \Omega_2 + \]

\[ + \sigma \Omega_3 + \lambda \Omega_4 + (\alpha - \bar{\beta} + 2 \pi + \bar{\tau}) \Omega_5 - (\varepsilon + \bar{\varepsilon} - \rho + 2 \rho) \Omega_6 - \kappa \Omega_7, \]  

(7)

\[ 2 \psi_3 = 3 \Delta \Omega_2 + \delta \Omega_3 - 3 \delta \Omega_6 - D \Omega_7 - 6 \nu \Omega_1 + 3 (\bar{\mu} - \mu - \bar{\gamma} + \gamma) \Omega_2 + (-\bar{\alpha} + 3 \beta + 3 \tau + \]

\[ + \bar{\pi}) \Omega_3 + 6 \lambda \Omega_5 + 3 (\alpha - \bar{\beta} + \pi + \bar{\tau}) \Omega_6 - (3 \varepsilon + \bar{\varepsilon} - \rho + 3 \rho) \Omega_7, \]

\[ \psi_4 = 2 [\Delta \Omega_3 - \delta \Omega_7 - 3 \nu \Omega_2 + (\bar{\mu} + 3 \gamma - \bar{\gamma}) \Omega_3 + 3 \lambda \Omega_6 + (-3 \alpha - \bar{\beta} + \bar{\tau}) \Omega_7], \]

we find the general solution:

\[ \Omega_0 = \Omega_7 = -q \frac{\pi}{4}, \quad \Omega_4 = q \Omega_3 = -\frac{\rho}{4}, \quad \Omega_1 = q \Omega_6 = -\left(\frac{\varepsilon + \bar{\rho}}{12}\right), \quad \Omega_2 = \Omega_5 = -\left(\frac{\rho}{3} + \frac{\pi}{12}\right). \]  

(8)

The Lanczos potential in Kerr geometry obtained in [12] is a special case of (8).

In (8) we observe that the NP components of $K_{\mu \nu \alpha}$ are linear combinations of the spin coefficients, as in the cases O, N and III studied in [8]. We conjecture that the Lanczos scalars $\Omega_r$ always can be chosen, for an adequate null tetrad, as a linear combination of the corresponding twelve spin coefficients.

References