Four-dimensional Tensor Identities

H. Torres-Silva, G. Posadas-Durán, & J. López-Bonilla

1Universidad de Tarapacá, EIEE, Arica, Chile
2ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Lindavista 07738, CDMX, México

Abstract
We show that an identity of Lovelock, for the conformal tensor in four dimensions, allows to motivate the Edgar’s identity which is important in the deduction of the wave equation for the Lanczos spintensor.

Keywords: Weyl tensor, Lanczos potential, tensor identities, four-space.

1. Introduction

The wave equation for the Lanczos generator [1-3] is important in general relativity [4, 5], and in its deduction participates the following Edgar’s identity [6]:

\[ L^{\mu\rho\lambda} W_{\mu\rho\lambda[\alpha} g_{\beta]} \nu + 2 W_{\mu\nu\rho[\alpha} L_{\beta]}^{\mu\rho} + \frac{1}{2} L^{\mu\rho} v W_{\mu\rho\alpha\beta} = 0, \]  

valid only in four dimensions, for arbitrary tensors verifying the properties:

\[ L_{\mu\nu\alpha} = -L_{\nu\mu\alpha}, \quad L_{\mu\nu\alpha} + L_{\nu\alpha\mu} + L_{\alpha\mu\nu} = 0, \quad L_{\mu\nu} v = 0, \quad W^{\mu}_{\nu\alpha\mu} = 0, \]  

\[ W_{\mu\nu\alpha\beta} = -W_{\nu\mu\alpha\beta} = -W_{\mu\nu\beta\alpha}, \quad W_{\mu\nu\alpha\beta} + W_{\mu\alpha\beta\nu} + W_{\mu\beta\nu\alpha} = 0, \]  

also satisfied by the Lanczos potential [1] and the conformal tensor [7].

Edgar [6] showed his identity employing the generalized Kronecker delta [8]; in Sec. 2 we use a result of Lovelock [9] to motivate the expression (1).

* Correspondence: J. López-Bonilla, ESIME-Zacatenco-IPN, Edif. 5, Col. Lindavista CP 07738, CDMX, México
E-mail: jlopezb@ipn.mx
2. Edgar’s identity

Lovelock [9] obtained the following four-dimensional identity:

\[ W_{[ab}^{[cd}\delta^q_p]} = 0, \]  

that is:

\[ \delta^q_a W_{b]p}^{cd} + \delta^c_a W_{b]p}^{dq} - \delta^d_a W_{b]p}^{cq} + \delta^c_p W_{ab}^{d]q} + \frac{1}{2} \delta^q_p W_{ab}^{cd} = 0; \]  

now we multiply (4) by \( W_{cd}^{pr} \) to deduce the relation:

\[ W_{\mu\nu\alpha\beta} W^{\mu\nu\tau\lambda} + 2 W_{\mu\nu\lambda} W^{\mu\nu\gamma} [\alpha \delta^\gamma_{\beta}] + 4 W^{\lambda}_{\mu\nu[\alpha} W_{\beta]}^{\mu\nu} = 0, \]  

where \( W_{\mu\nu\alpha\beta} \) has all symmetries of the Weyl tensor.

If in (5) we contract \( \alpha \) with \( \lambda \), we obtain the Lanczos identity [10-12]:

\[ W_{abcd} W^{abcd} = \frac{1}{4} W_2 \delta^r_a, \quad W_2 \equiv W_{\mu\nu\alpha\beta} W_{\mu\nu\alpha\beta}, \]  

hence (5) acquires the form [13]:

\[ W_{\mu\nu\alpha\beta} W^{\mu\nu\tau\lambda} - 4 W_{\mu[\tau\nu} W_{\beta]}^{\mu\lambda} - \frac{1}{4} W_2 \delta^\tau_\alpha \delta^\lambda_\beta = 0. \]  

Finally, the Edgar’s identity (1) is immediate from (5) if we multiply it by an arbitrary vector \( A_\lambda \) and we introduce the tensor:

\[ L^{\mu\nu\tau\lambda} \equiv W^{\mu\nu\tau\lambda} A_\lambda, \]  

which verifies the properties (2).

Received February 6, 2017; Accepted March 4, 2017

References


