Killing Magnetic Curves in Three Dimensional Isotropic Space

Alper O. Ögrenmiş

Department of Mathematics, Faculty of Science, Firat University, Elazığ, 23119, Turkey

Abstract

In this paper, we study and classify the magnetic curves in the isotropic 3-space associated to a Killing vector field $V = v_i \partial_i$ with $\partial_i = \frac{\partial}{\partial x_i}$ and $v_i \in \mathbb{R}$, $i = 1, 2, 3$.

Keywords: Magnetic trajectory, Lorentz force, killing vector field, isotropic space.

1 Introduction

Let $(N, g)$ denote a Riemannian manifold and $F$ a closed 2-form. $F$ is said to be a magnetic field. The Lorentz force of a magnetic background $(M, g, F)$ is the skew symmetric $(1, 1)$-type tensor field $\phi$ on $N$ satisfying

$$g(\phi(X), Y) = F(X, Y)$$

for any $X, Y$ tangent to $N$. Hence a magnetic curve associated to $F$ is a smooth curve $c$ on $N$ satisfying

$$\nabla_c c' = \Phi(c'),$$

where $\nabla$ is the Levi-Civita connection of $g$. The equation (1.2) is known as the Lorentz equation.

Since $F$ is skew symmetric in (1.1), the magnetic curve $c$ has constant speed, i.e. $g(c', c') = \lambda = \text{const}$. In the particular case $\lambda = 1$, it is called a normal magnetic curve. We consider only the normal magnetic curves all over this paper.

The first study of magnetic fields was treated on Riemannian surfaces (see e.g. [7, 25]), then in 3-dimensional context, on $\mathbb{E}^3$ [15], $\mathbb{E}_1^3$ [16], $S^3$ [8], $S^2 \times \mathbb{R}$ [23] etc. For more study of the magnetic curves on the (semi-) Riemannian manifolds, we refer to [11, 27, 10, 12, 13, 17, 21, 22].

On the other hand, the isotropic 3-space $I^3$ is a Cayley–Klein space defined from a 3-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure which is an ordered triple $(w, f_1, f_2)$, where $w$ is a plane in $P(\mathbb{R}^3)$ and $f_1, f_2$ are two complex-conjugate straight lines in $w$, see [13]–[20].

The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the absolute plane $w$ is given by $X_0 = 0$ and the absolute lines $f_1, f_2$ by $X_0 = X_1 + iX_2 = 0$, $X_0 = X_1 - iX_2 = 0$. The intersection point $F(0:0:1)$ of these two lines is called the absolute point. The affine coordinates are obtained by $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, $x_3 = \frac{X_3}{X_0}$.

In this paper, our aim is to classify the magnetic curves in $I^3$. In this manner we derive some classifications for the magnetic curves and $N$–magnetic curves with constant curvature (see Definition 4.1) associated to the Killing vector field $V = v_1 \partial_1 + v_2 \partial_2 + v_3 \partial_3$ in $I^3$, where $\partial_i = \frac{\partial}{\partial x_i}$ are orthonormal basis vector fields, $i = 1, 2, 3$.

1 Correspondence: E-mail: aogrenmis@firat.edu.tr
2 Preliminaries

In this section we provide the fundamental notions on isotropic spaces from (3, 4), (26)- (31).

The isotropic distance in \( I^3 \) of two points \( x = (x_i) \) and \( y = (y_i), i = 1, 2, 3, \) is defined as
\[
\|x - y\|_I = \sqrt{\sum_{j=1}^{2}(y_j - x_j)^2}.
\] (2.1)

The lines in \( x_3 \)-direction are called isotropic lines. The plane containing an isotropic line is said to be an isotropic plane. Other planes are non-isotropic.

The isotropic scalar product between two vectors \( a = (a_i) \) and \( b = (b_i) \) in \( I^3 \) is given by
\[
\langle a, b \rangle_I = \begin{cases} a_1 b_1 + a_2 b_2, & a_i \neq 0 \text{ or } b_i \neq 0, \ (i = 1, 2), \\ a_3 b_3, & a_i = b_i = 0, \ (i = 1, 2). \end{cases}
\] (2.2)

The cross product of two vectors \( a = (a_i) \) and \( b = (b_i) \) in \( I^3 \) can be defined by
\[
a \times_I b = \begin{vmatrix} e_1 & e_2 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\] (2.3)
for \( e_1 = (1, 0, 0), \ e_2 = (0, 1, 0) \). It is easy to check that
\[
\langle a \times_I b, c \rangle_I = \det (a, b, \overline{c}),
\]
where \( \overline{c} \) means the projection of \( c \) on the Euclidean \((x_1, x_2)\)-plane. For more details, see (5).

Let \( c : I \rightarrow I^3, I \subset \mathbb{R}, \) be a curve parameterized by the arc length. It is called admissible if it has no tangent vector field in \( x_3 \)-direction. An admissible curve can be given in the form
\[
c(s) = (x(s), y(s), z(s)), \ \overline{c}(s) = (x(s), y(s)),
\]
where \( \overline{c}' = (x'(s), y'(s)) \neq 0 \). The curvature \( \kappa(s) \) and the torsion \( \tau(s) \) are respectively defined by
\[
\kappa(s) = \frac{\det(\overline{c}'(s), \overline{c}''(s))}{\kappa^2(s)}, \ \tau(s) = \frac{\det(c'(s), c''(s), c'''(s))}{\kappa^2(s)}, \ \kappa(s) \neq 0,
\]
and the associated trihedron is given by
\[
\begin{align*}
T &= (x'(s), y'(s), z'(s)), \\
N &= \frac{1}{\kappa(s)}((x''(s), y''(s), z''(s))), \\
B &= (0, 0, 1).
\end{align*}
\] (2.4)

For such vector fields the following Frenet’s formulas hold
\[
T' = \kappa N, \ N' = -\kappa T + \tau B, \ B' = 0.
\] (2.5)
3 Killing Magnetic Trajectories in $\mathbb{I}^3$

Let $(N, g)$ be a Riemannian manifold and $X$ a vector field on $N$. If $L_X g = 0$ then $X$ is called a Killing vector field, where $L$ denotes the Lie derivative with respect to $X$. It is easily seen that $X$ is a Killing vector field on $N$ if and only if

$$g (\nabla_Y X, Z) + g (\nabla_Z X, Y) = 0,$$

where $\nabla$ is the Levi-Civita connection of $g$.

The 2-forms on 3-dimensional manifolds may correspond to the vector fields via the Hodge $\star$ operator and the volume form $dv_g$ of the manifold. Hence, we can consider the Killing magnetic fields associated the Killing vector fields.

Note that the cross product of any vector fields $V, X, Y, Z$ is not identically zero.

Let $\Phi (V) = V \times V.$

From (1.2) and (3.1), the Lorentz force of $F_V$ is (see e.g. [8, 13])

$$(3.1)$$

From (1.2) and (3.1), the Lorentz force of $(\mathbb{I}^3, \langle \cdot, \cdot \rangle, F_V)$ is

$$(3.2)$$

We call $c$ Killing magnetic curve.

Now let $c$ be a curve in $\mathbb{I}^3$, parametrized by the arc length and given in the coordinate form

$$c (s) = (x(s), y(s), z(s)), \quad s \in I \subset \mathbb{R},$$

where $x, y$ and $z$ are smooth functions satisfying the initial conditions:

$$x(0) = x_0, \quad x'(0) = X_0, \quad y(0) = y_0, \quad y'(0) = Y_0 \quad \text{and} \quad z(0) = z_0, \quad z'(0) = Z_0.$$  

Remark 3.1. $c$ is a non-isotropic line in $\mathbb{I}^3$ when $V \equiv 0$. Afterwards we assume that $V$ is not identically zero.

By the following result, we classify the normal magnetic trajectories associated to the Killing vector $V = v_1 \partial_1 + v_2 \partial_2 + v_3 \partial_3$ in $\mathbb{I}^3$, where $\partial_i = \frac{\partial}{\partial x_i}$ and $v_1, v_2, v_3 \in \mathbb{R}$.

Theorem 3.1. Let $c$ be a normal magnetic curve associated to the Killing vector $V = v_1 \partial_1 + v_2 \partial_2 + v_3 \partial_3$ in $\mathbb{I}^3$ with the initial conditions (3.4). Then:

(i) If $V = (v_1, v_2, 0)$,

$$c (s) = \left( v_2 Z_0 s^2 + X_0 s + x_0, \quad - \frac{v_1}{2} Z_0 s^2 + Y_0 s + y_0, \quad Z_0 s + z_0 \right);$$  

(ii) if $V = (v_1, v_2, v_3 \neq 0)$,

$$c (s) = \left( \lambda_1 - x_0 \cos (v_3 s) + (\lambda_2 - y_0) \sin (v_3 s) + \frac{Z_0 v_1}{v_3} s + \lambda_1, \right.$$  

$$\left( \lambda_2 - y_0 \cos (v_3 s) - (\lambda_1 - x_0) \sin (v_3 s) + \frac{Z_0 v_2}{v_3} s + \lambda_2, \right.$$  

$$\left( Z_0 s + z_0 \right).$$
where $\lambda_1 = x_0 + \frac{Y_0 - Z_0 v_1}{v_3}$ and $\lambda_2 = y_0 + \frac{X_0 - Z_0 v_2}{v_3}$.

Proof. If $c$ is a normal magnetic trajectory in $I^3$, then it is a solution of (3.2). We have to consider two cases for the proof.

Case 1. $V = (v_1, v_2, 0)$. Then it follows from (2.3) and (3.2) that
\[
\begin{cases}
  x'' = v_2 z', \\
  y'' = -v_1 z', \\
  z'' = 0, z = Z_0 s + z_0.
\end{cases} \tag{3.7}
\]
After considering the initial conditions (3.4) into (3.7), $c$ derives the form (3.5). This implies the statement (i).

Case 2. $V = (v_1, v_2, v_3 \neq 0)$. Hence, from (2.4) and (3.2) we have
\[
\begin{cases}
  x'' = -Z_0 v_2 + v_3 y', \\
  y'' = Z_0 v_1 - v_3 x'.
\end{cases} \tag{3.8}
\]
We may formulate the Cauchy problem associated to system (3.8) and the initial conditions (3.4) as follows:
\[
\begin{cases}
  x''' = Z_0 v_1 v_3 - v_3^2 x', \\
  y''' = Z_0 v_2 v_3 - v_3^2 y'.
\end{cases} \tag{3.9}
\]
Solving (3.9), we obtain
\[
x(s) = \frac{Z_0 v_1}{v_3} s + \frac{X_0 - Z_0 v_1}{v_3} \sin (v_3 s) + \frac{Y_0 - Z_0 v_2}{v_3} \cos (v_3 s) + \]
\[
x_0 + \frac{Y_0 - Z_0 v_1}{v_3},
\]
\[
y(s) = \frac{Z_0 v_2}{v_3} s - \frac{Y_0 - Z_0 v_2}{v_3} \sin (v_3 s) + \frac{X_0 - Z_0 v_2}{v_3} \cos (v_3 s) + \\
y_0 + \frac{X_0 - Z_0 v_2}{v_3},
\]
which completes the proof.

4 $N$–Magnetic and $B$–Magnetic Curves in $I^3$

Definition 4.1. Let $c : I \subset \mathbb{R} \rightarrow M$ be a curve in an oriented 3-dimensional Riemannian manifold $(M, g)$ and $F$ be a magnetic field on $M$. The curve $c$ is an $N$–magnetic curve (respectively $B$–magnetic curve) if the normal vector field $N$ (respectively the binormal vector field $B$) of the curve satisfies the Lorentz force equation, i.e., $\nabla_t N = \Phi (N) = V \times N$ (respectively $\nabla_t B = \Phi (B) = V \times B$).

In this section in order to obtain certain results, we assume that the curve $c$ has nonzero constant curvature $\kappa_0$.

Let consider the arc length curve $c$ in $I^3$, parametrized by
\[
c(s) = (x(s), y(s), z(s)), \tag{4.1}
\]
where 

\[ N = \frac{1}{\kappa_0} (x''(s), y''(s), z''(s)). \]

**Case 1.** \( v_3 = 0 \). By using the cross product in \( \mathbb{I}^3 \) and (4.3), we get

\[
\begin{cases}
  x''' = v_2 z'', \\
  y''' = -v_1 z'', \\
  z''' = 0, \quad z = \frac{W_0}{2v_3} s^2 + Z_0 s + z_0.
\end{cases}
\]  

By solving the Cauchy problem associated to system (4.4) and the initial conditions (4.2), we obtain

\[
\begin{align*}
  x(s) &= \frac{W_0v_2}{6} s^3 + \frac{T_0}{2} s^2 + X_0 s + x_0, \\
  y(s) &= -\frac{W_0v_1}{6} s^3 + \frac{U_0}{2} s^2 + Y_0 s + y_0,
\end{align*}
\]

which implies the first statement.

**Case 2.** \( v_3 \neq 0 \). From (2.3) and (4.3), we derive

\[
\begin{cases}
  x''' = v_2 z'' - v_3 y'', \\
  y''' = -v_1 z'' + v_3 x''.
\end{cases}
\]
By solving the Cauchy problem associated to system (4.5) and the initial conditions (4.2), we obtain

\[
x(s) = \frac{v_1 W_0}{2v_3} s^2 + (X_0 - U_0) s - T_0 \cos(v_3 s) + U_0 \sin(v_3 s) + x_0 + T_0 \\
y(s) = \frac{v_3 W_0}{2v_3} s^2 + (Y_0 + T_0) s - U_0 \cos(v_3 s) - T_0 \sin(v_3 s) + y_0 + U_0,
\]

which completes the proof.

On the other hand, since the binormal vector field \(B\) of a curve in \(I^3\) is completely isotropic vector, i.e. \((0, 0, 1)\), the corresponding Lorentz equation becomes via Definition 4.1 as follows

\[0 = V \times_B B.\]

This implies that either \(V = 0\) or \(V\) is completely isotropic vector, i.e. \((0, 0, v_3)\). Thus we obtain the following result:

**Corollary 4.2.** All curves in \(I^3\) are \(B\)--magnetic curves associated to the Killing vector \(V = (0, 0, v_3)\).

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**References**


